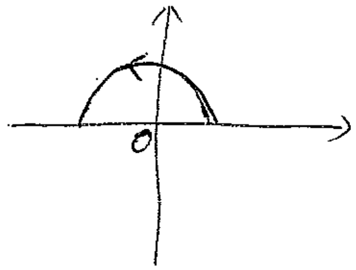
Homework 3.

$\gamma = e^{it}, \quad 0 \le t \le \pi$



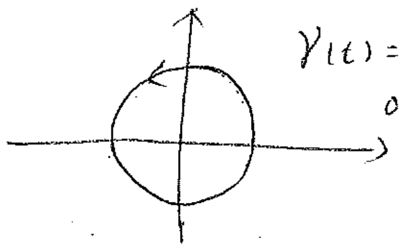
1. (1) $\displaystyle\int_\gamma \operatorname{im}(z)\, dz.$

$$= \int_0^{\pi} \sin t \; i e^{it}\, dt$$

$$= i\int_0^{\pi} \sin t \cos t \, dt - \int_0^{\pi} \sin^2 t \, dt$$

$$= -\frac{\pi}{2}$$

$\gamma(t) = e^{it}$
$0 \le t \le 2\pi$

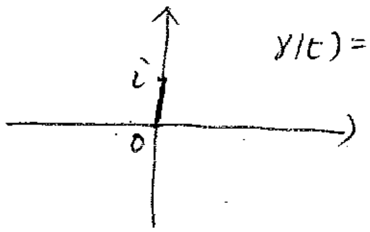


(2) $\displaystyle\int_\gamma \bar{z}\, dz$

$$= \int_0^{2\pi} e^{-it} \cdot i e^{it} \, dt$$

$$= 2\pi i$$

$\gamma(t) = ti, \quad 0 \le t \le 1$



(3) $\displaystyle\int_\gamma (\operatorname{Re}(z)^2 - \operatorname{im}(z)^2)\, dz$

$$= \int_0^1 (-t^2) \cdot i \, dt$$

$$= -\frac{1}{3} i$$

2. Proof: For any given simple closed curve γ in G which contains the point a. Let R be the number such that the closed disk $\overline{D}(a,r) \subseteq I(\gamma)$ for any $0<r \le R$, and $\overline{D}(a,r) \subseteq C$

By deformation theorem, we have

$$\int_\gamma f(z)\,dz = \int_{\gamma(a,r)} f(z)\,dz, \quad \text{where } 0<r\le R.$$

$$= \int_0^{2\pi} f(a+re^{it})\cdot rie^{it}\,dt$$

By assumption, $|f(z)|\le M$, i.e. $|f(a+re^{it})|\le M$

for $z\in \overline{D}(a,r)\subseteq I(\gamma)\cap D$.

By the estimation theorem, we get

$$\left|\int_\gamma f(z)\,dz\right| \le \int_0^{2\pi} |f(a+re^{it})|\,|rie^{it}|\,dt$$

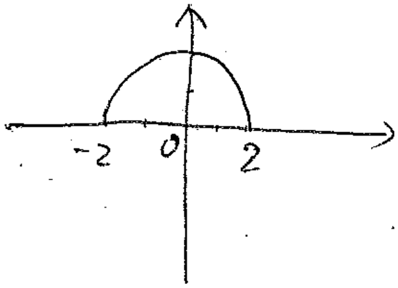
$$\le 2\pi r M$$

Letting $r\to 0$, we get

$$\int_\gamma f(z)\,dz = 0.$$

3. (1)

$$\int_\gamma (z^3+3)\,dz$$

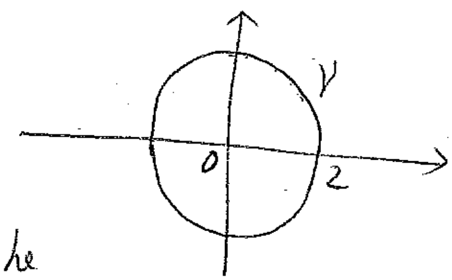


$$= \int_0^\pi (8e^{3it}+3)\cdot 2ie^{it}\,dt \qquad \gamma(t) = 2e^{it}, \quad 0 \le t \le \pi$$

$$= 16i\int_0^\pi e^{4it}\,dt + 6i\int_0^\pi e^{it}\,dt$$

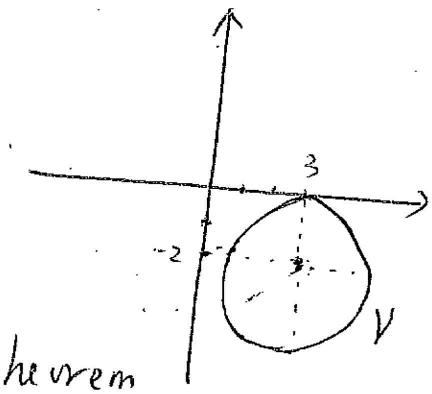
$$= 12i^2 = -12$$

(2) since the function $f(z)=z^3+3$ is holomorphic on and inside the given circle $\gamma(0,2)$, using Cauchy's theorem, we get



$$\int_\gamma (z^3+3)\,dz = 0$$

(3) Since $e^{\frac{1}{z}}$ is holomorphic on and inside the circle $\gamma(3-2i, 2)$, So $\int_\gamma e^{\frac{1}{z}}\,dz = 0$ by Cauchy's theorem.



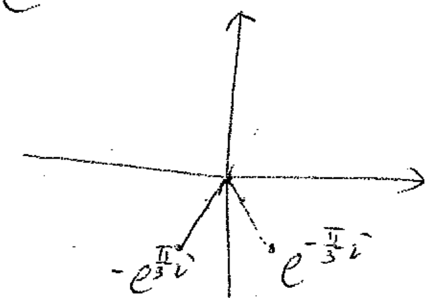
4. $$\int_\gamma \frac{dz}{z^2+z+1} = \int_\gamma \frac{dz}{(z+\frac{1+\sqrt{3}i}{2})(z-\frac{1-\sqrt{3}i}{2})}$$

$$= \int_\gamma \frac{1}{\sqrt{3}i}\left(\frac{1}{z - \frac{1-\sqrt{3}i}{2}} - \frac{1}{z + \frac{1+\sqrt{3}i}{2}} \right) dz$$

$$= \frac{1}{\sqrt{3}\,i}\left\{ \int_\gamma \frac{1}{z - \frac{1-\sqrt{3}i}{2}}\,dz - \int_\gamma \frac{1}{z + \frac{1+\sqrt{3}i}{2}}\,dz \right\}$$

$$\frac{1-\sqrt{3}i}{2} = e^{-\frac{\pi}{3}i}, \quad \frac{1+\sqrt{3}i}{2} = e^{\frac{\pi}{3}i}$$

Let $a = e^{-\frac{\pi}{3}i}$, $b = -e^{\frac{\pi}{3}i}$



We will divide four cases to consider this problem.

Case 1: $a, b \in O(\gamma)$. under this case, the function $\frac{1}{z^2+z+1}$ is holomorphic on and inside of γ. by Cauchy's theorem, we have $\int_\gamma \frac{dz}{z^2+z+1} = 0$

Case 2: $a \in I(\gamma)$, $b \in O(\gamma)$.

By Cauchy's theorem, $\int_\gamma \frac{1}{z + \frac{1+\sqrt{3}i}{2}}\,dz = 0$

By deformation theorem, $\int_\gamma \frac{1}{z - \frac{1-\sqrt{3}i}{2}}\,dz = \int_{\gamma(\frac{1-\sqrt{3}i}{2},\, r)} \frac{1}{z - \frac{1-\sqrt{3}i}{2}}\,dz$,

$= 2\pi i$ by the fundamental integral.

where r is a number such that $\overline{D}(\frac{1-\sqrt{3}i}{2}, r) \subseteq I(\gamma)$

Therefore,
$$\int_\gamma \frac{1}{z^2+z+1}\,dz = \frac{1}{\sqrt{3}i}\cdot\int_\gamma \frac{1}{z-\frac{1-\sqrt{3}i}{2}}\,dz$$
$$= \frac{1}{\sqrt{3}i}\cdot 2\pi i$$
$$= \frac{2\sqrt{3}\pi}{3}$$

Case 3: $a \in O(\gamma)$, $b \in I(\gamma)$. Similar reason to the Case 2. We get $\int_\gamma \frac{1}{z-\frac{1-\sqrt{3}i}{2}}\,dz = 0$ and $\int_\gamma \frac{1}{z+\frac{1+\sqrt{3}i}{2}}\,dz = 2\pi i$

Therefore,
$$\int_\gamma \frac{1}{z^2+z+1}\,dz = -\frac{1}{\sqrt{3}i}\int_\gamma \frac{1}{z+\frac{1+\sqrt{3}i}{2}}\,dz$$
$$= -\frac{2\sqrt{3}}{3}\pi$$

Case 4: $a, b \in I(\gamma)$. By the deformation theorem, there exists $r_1 > 0$, $r_2 > 0$ such that $\overline{D}(\frac{1-\sqrt{3}i}{2}, r_1) \subseteq I(\gamma)$ and $\overline{D}(-\frac{1+\sqrt{3}i}{2}, r_2) \subseteq I(\gamma)$. So

$$\int_\gamma \frac{1}{1+z+z^2}\,dz = \frac{1}{\sqrt{3}i}\left(\int_\gamma \frac{1}{1-\frac{1-\sqrt{3}i}{2}}\,dz - \int_\gamma \frac{1}{z+\frac{1+\sqrt{3}i}{2}}\,dz\right)$$

$$= \frac{1}{\sqrt{3}i}\left(\int_{\gamma\left(\frac{1-\sqrt{3}i}{2},\, r_1\right)} \frac{1}{1-\frac{1-\sqrt{3}i}{2}}\,dz - \int_{\gamma\left(-\frac{1+\sqrt{3}i}{2},\, r_2\right)} \frac{1}{1+\frac{1+\sqrt{3}i}{2}}\,dz\right)$$

Fundamental integral $\cong \frac{1}{\sqrt{3}i}\left(2\pi i - 2\pi i\right)$

$= 0.$