Assignment 1.

1. In general, $z^2 \neq |z|^2$. For example, take $z = i \in \mathbb{C}$

then $z^2 = -1$ and $|z|^2 = 1$. Thus $z^2 \neq |z|^2$.

Let $z = x + iy$, $x, y \in \mathbb{R}$. Then

$$z^2 = x^2 + 2xyi - y^2 \quad \text{and} \quad |z|^2 = x^2 + y^2$$

If $z^2 = |z|^2$, we have

$$x^2 + 2xyi - y^2 = x^2 + y^2.$$

i.e. $\begin{cases} x^2 - y^2 = x^2 + y^2 & \Rightarrow y = 0 \\ 2xy = 0 & \Rightarrow x = 0 \text{ or } y = 0 \end{cases}$

Therefore, we get $z^2 = |z|^2$ iff $y = 0$, i.e. $z \in \mathbb{R}$.

2. Riemann sphere is defined by

$$\Sigma := \{(x, y, u) \in \mathbb{R}^3,\ x^2 + y^2 + (u - \tfrac{1}{2})^2 = \tfrac{1}{4}\} \quad (1)$$

We know that there is a natutural correspondene between

$\tilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and Σ, which is given by

$$z = x + iy \in \mathbb{C} \longleftrightarrow z' = \left(\frac{x}{1+r^2}, \frac{y}{1+r^2}, \frac{r^2}{1+r^2}\right) \in \Sigma \quad (2)$$

$r = x^2 + y^2$

$$\infty \longleftrightarrow (0, 0, 1) \in \Sigma$$

The real axis of complex plane, which is denoted as X, given by

$$X = \{ \cancel{z} = x + iy \in \mathbb{C} : y = 0 \} \quad (3)$$

The Imaginary axis of complex plane, which is denoted as Y, given by

$$Y = \{ z = x + iy \in \mathbb{C} : x = 0 \} \quad (4)$$

Combining (1), (2) and (3), (4) we get

The subsets of the Riemann sphere corresponding to the real and the imaginary axis of the complex plane $\mathbb{C}$, which are denoted as Σ_X and Σ_Y respectively, are the following:

$$\Sigma_X = \{ (x, 0, u) \in \mathbb{R}^3 : x^2 + (u - \tfrac{1}{2})^2 = \tfrac{1}{4} \}$$

$$= \{ (x, y, u) \in \mathbb{R}^3 : x^2 + (u - \tfrac{1}{2})^2 = \tfrac{1}{4}, y = 0 \}$$

$$= \{ (\frac{x}{1+x^2}, 0, \frac{x^2}{1+x^2}) \in \mathbb{R}^3, x \in \mathbb{R} \}$$

$$\Sigma_Y = \{ (\cancel{x}, y, u) \in \mathbb{R}^3 : y^2 + (u - \tfrac{1}{2})^2 = \tfrac{1}{4} \}$$

$$= \{ (x, y, u) \in \mathbb{R}^3 : x^2 + y^2 + (u - \tfrac{1}{2})^2 = \tfrac{1}{4}, x = 0 \}$$

$$= \{ (0, \frac{y}{1+y^2}, \frac{y^2}{1+y^2}) \in \mathbb{R}^3, y \in \mathbb{R} \}$$

3. (i) Let z be the fixed point of f, then by the definition of f and fixed point, we have

$$f(z) = \frac{az+b}{cz+d} = z, \text{ where } ad-bc \neq 0.$$

i.e. we have $$cz^2 + (d-a)z - b = 0 \quad (*)$$

Case 1: If $c \neq 0$, then equation $(*)$ is a quadratic equation. It has one (repeated) or two distict roots. correspondingly, f has one or two fixed points.

Case 2: If $c=0$, $d-a \neq 0$, then $(*)$ is a linear equation. It has one solution: $z = \frac{b}{d-a}$. In this case f has one fixed point.

Case 3: If $c=0$, $d-a=0$ $(a=d\neq 0)$, then in this case $b \neq 0$. Otherwise, f is the identity map. and

$$f(z) = \frac{az+b}{a} = z + \frac{b}{a}.$$

Sine f is a transformation from extended plane $\tilde{C} = C \cup \{\infty\}$ to $\tilde{C}$; f maps ∞ to ∞. So ∞ is the fixed point of f under this case.

In all, we proved that f has one or two fixed points.

(ii) By assumption, α and β are two distinct fixed points of f, then by (*) we have

$$c\alpha^2 + (d-a)\alpha - b = 0, \text{ i.e } c\alpha^2 - a\alpha = b - d\alpha \quad (1)$$

and $$c\beta^2 + (d-a)\beta - b = 0, \text{ i.e } c\beta^2 - a\beta = b - d\beta \quad (2)$$

By the definition of w, we get

$$\frac{w-\alpha}{w-\beta} = \frac{\frac{az+b}{cz+d} - \alpha}{\frac{az+b}{cz+d} - \beta} = \frac{az+b - \alpha(cz+d)}{az+b-\beta(cz+d)}$$

$$= \frac{z(a-c\alpha) + b - d\alpha}{z(a-c\beta) + b - d\beta}$$

(1)

$$= \frac{z(a-c\alpha) - \cancel{\alpha}\alpha(a - c\alpha)}{z(a-c\beta) - \cancel{\beta}\beta(a-c\beta)}$$

(2)

$$= \frac{(a-c\alpha)(z-\alpha)}{(a-c\beta)(z-\beta)}$$

$$= k\frac{z-\alpha}{z-\beta}, \quad \text{where } k := \frac{a-c\alpha}{a-c\beta}$$

(a) It's clear that $\left|\frac{z-a}{z-\beta}\right| = \lambda$ is a circline with inverse points α and β. Then f maps $\left|\frac{z-a}{z-\beta}\right| = \lambda$ to another circline with inverse points $f(\alpha)$ and $f(\beta)$.

Since α and β are two fixed points, i.e. $f(\alpha)=\alpha$, $f(\beta)=\beta$.

Thus ~~the circline under f~~

$$\left|\frac{f(z)-f(\alpha)}{f(z)-f(\beta)}\right| = \left|\frac{w-\alpha}{w-\beta}\right|,$$ which is exactly the image under f of the circline $\left|\frac{z-\alpha}{z-\beta}\right| = \lambda$.

By previous result, we have $\left|\frac{w-\alpha}{w-\beta}\right| = |K|\left|\frac{z-\alpha}{z-\beta}\right| = |K|\lambda$,

where $K = \frac{a-c\alpha}{a-c\beta}$.

(b) Since $\frac{w-\alpha}{w-\beta} = K\frac{z-\alpha}{z-\beta}$, then

$$\operatorname{arc}\arg\left(\frac{z-\alpha}{z-\beta}\right) = \operatorname{arc}\arg\left(\frac{1}{K}\frac{w-\alpha}{w-\beta}\right), \ \#1.$$

Sine $\arg\left(\frac{1}{K}\frac{w-\alpha}{w-\beta}\right) = \arg\left(\frac{w-\alpha}{w-\beta}\right) - \arg K$.

Thus the image of $\operatorname{arc}\arg(z-\alpha)/(z-\beta) = \mu \pmod{2\pi}$ is

$$\operatorname{arc}\arg\left(\frac{w-\alpha}{w-\beta}\right) = \mu \pmod{2\pi} + \arg K, \text{ where } K = \frac{a-c\alpha}{a-c\beta}.$$

(iii) α is a single fixed point of f. from $(*)$, we have

$$c\alpha^2 + (d-a)\alpha - b = 0, \text{ i.e } c\alpha^2 - a\alpha = b - d\alpha \quad (3)$$

and $\alpha = \dfrac{a-d}{2c}$, $\quad d + c\alpha = a - c\alpha \quad (4)$

By the representation of w, we get

$$\frac{1}{w-\alpha} = \frac{1}{\frac{az+b}{cz+d} - \alpha} = \frac{cz+d}{az+b-\alpha(cz+d)}$$

$$= \frac{cz+d}{(a-c\alpha)z + (b-d\alpha)}$$

$$= \frac{cz+d}{(a-c\alpha)z - (a-c\alpha)\alpha} \quad (3)$$

$$= \frac{cz+d}{(z-\alpha)(a-c\alpha)}$$

$$= \frac{c(z-\alpha) + d + c\alpha}{(z-\alpha)(a-c\alpha)}$$

$$= \frac{c}{a-c\alpha} + \frac{d+c\alpha}{(z-\alpha)(a-c\alpha)} \quad (4)$$

$$= \frac{c}{a-c\alpha} + \frac{1}{z-\alpha}$$

4. Let $S = \{z_0\}$. Then $S^c = \mathbb{C} \setminus S$. To prove $S = \{z_0\}$ is closed, it's enough to prove that S^c is open.

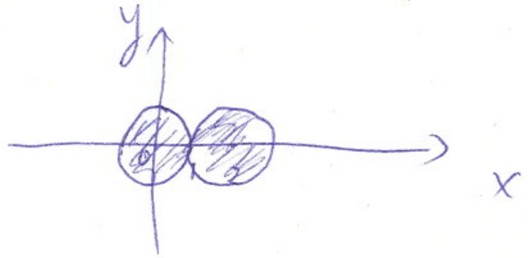
For any $z \in S^c = \mathbb{C} \setminus S$, then $z \neq z_0$ and thus $|z - z_0| > 0$. Take $r = \frac{1}{2}|z - z_0|$, then $D(z, r) \subseteq S^c$. By the definition of open set, we prove that S^c is open. so $S = \{z_0\}$ is closed.

Let T be the set composed of finite number of points in $\mathbb{C}$. Denote $T = \{t_1, t_2 \dots t_n\}$. Then $T = \bigcup_{i=1}^{n} \{T_i\}$.

By previous result, $\{T_i\}$ is closed. so the finite union of closed sets is closed, ie. T is closed. So T^c is open

This proves that the complement of any finite number of points in $\mathbb{C}$ is an open set.

5. $X = \{z \in \mathbb{C} : |z| \leq 1\} \cup \{z \in \mathbb{C} : |z - 2| \leq 1\}$

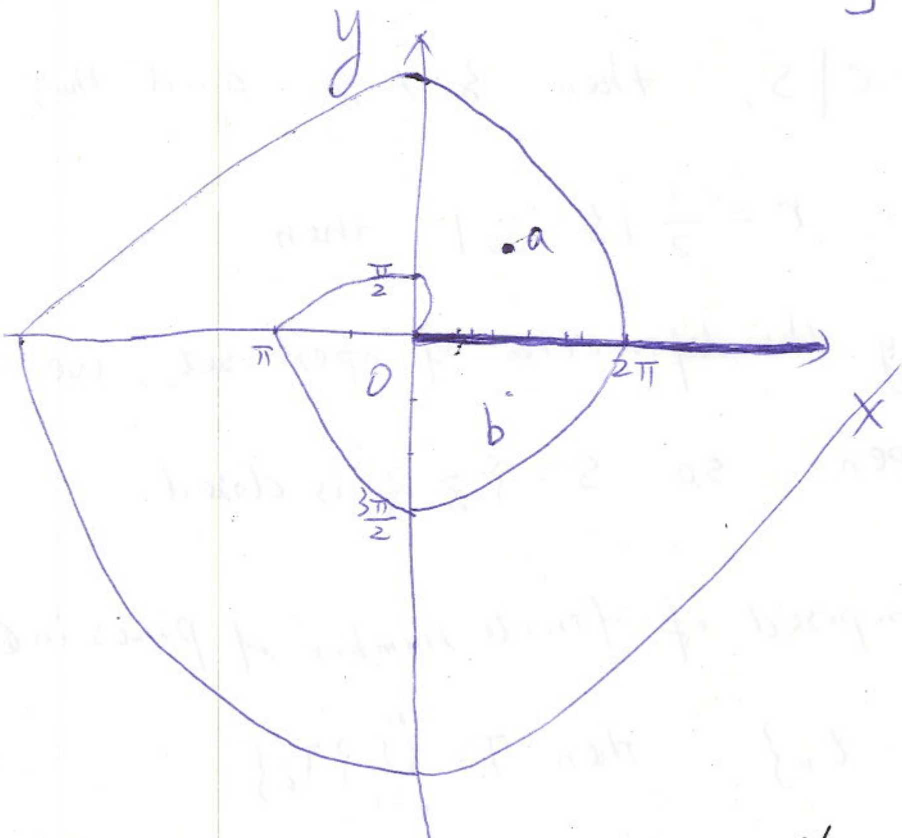


connected.

$X = \mathbb{C} \setminus A \cup B$, where $A = \{ z \in \mathbb{C} : \operatorname{Re} z \in (0, \infty) \text{ and } \operatorname{im} z = 0 \}$.

$B = \{ z = \theta e^{i\theta}, \ 0 \le \theta < \infty \}$

NOT connected



Each " circuit " of the spiral is a component:

$$\left\{ re^{i\theta} : \begin{array}{l} 2n\pi < \theta < 2(n+1)\pi, \\ 2n\pi < r < 2(n+1)\pi \end{array} \right\}$$