Priestley Chapter 10 Some Solutions

10.1 (i) $\displaystyle\int_\gamma z^2\,dz, \qquad \gamma(t) = e^{it}, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$$= \int_{t=-\pi/2}^{\pi/2} (e^{it})^2\, i e^{it}\, dt$$

$$= \frac{1}{3}\Big[e^{3it}\Big]_{-\pi/2}^{\pi/2} = \frac{1}{3}\cdot 2i \sin\left(\frac{3\pi}{2}\right) = -\frac{2}{3}i.$$

(ii) $\displaystyle\int_\gamma \mathrm{Re}(z)\,dz \qquad \gamma(t) = t + it^2, \quad 0 \le t \le 1.$

$$= \int_{t=0}^{1} \mathrm{Re}(t+it^2)\,(1+2it)\,dt = \int_0^1 (t+2it^2)\,dt$$

$$= \frac{1}{2} + \frac{2}{3}i.$$

(iii) $\displaystyle\int_\gamma \frac{1}{z}\,dz \qquad \gamma(t) = e^{-it}, \quad 0 \le t \le 8\pi$

$$= \int_0^{8\pi} e^{it}(-i)e^{-it}\,dt = -i8\pi = -2\pi i \times 4.$$

(iv) $\displaystyle\int_\gamma e^z\,dz \qquad \gamma$ the join of $[0,1], [1,1+i], [1+i, i]$

$$= \int_0^1 e^t\,dt + \int_0^1 e^{1+t(i-1)}(i-1)\,dt + \int_0^1 e^{1+i-t}(-1)\,dt$$

$$= -1 + 2e^i - e^{1+i}$$

(v) $\int_\gamma |z|^4\, dz \qquad \gamma = [-1+i, 1+i]$

$$= \int_0^1 \sqrt{(-1+2t)^2+1^2}^{\,4}\; 2\,dt = \cancel{\tfrac{4}{3}} \frac{92}{5}.$$

<u>10.2</u> $\int_{\gamma(0;1)} f(z)\, dz. \qquad \gamma(t) = e^{it},\ 0 \le t \le 2\pi.$

(i) $\int_\gamma |z^4|\, dz = \int_0^{2\pi} 1^4\, i e^{it}\, dt = 0.$

(ii) $\int_\gamma (\mathrm{Re}(z))^2\, dz = \int_0^{2\pi} \cos^2(e^{it})\, i e^{it}\, dt$

$$= \left[e^{it} - \tfrac{1}{2}\sin(2e^{it}) \right]_0^{2\pi} = 0.$$

(iii) $\int_\gamma \frac{z^4-1}{z^2}\, dz = \int \frac{e^{4it}-1}{e^{2it}}\, i e^{it}\, dt$

$$= i \int_0^{2\pi} (e^{3it} - e^{-it})\, dt = 0.$$

(iv) $\int_\gamma \sin(z)\, dt = \int_0^{2\pi} \sin(e^{it})\, i e^{it}\, dt = 0.$