Math 3TP3 Truth and Provability Term 1 Autumn 2014–2015
Assignment 3 due 17 October 2014

1) Prove clauses O5 and O7 of Theorem 11.1 that Q is order adequate.

   O5: For any n, if $Q \vdash \varphi(0)$ or $Q \vdash \varphi(\bar{1})$ or $\cdots Q \vdash \varphi(\bar{n})$ then $Q \vdash \exists x \le \bar{n}\, \varphi(x)$

   O7: For any n, $Q \vdash \forall x\, (\bar{n} \le x \to (\bar{n} = x \vee S\bar{n} \le x))$

2) Prove clauses 3, 5, 8 of Theorem 12.1. That is, the following statements are provable in $\mathrm{I}\Delta_0$.

   3): $\forall x \forall y\, (Sx + y = S(x+y))$

   5): $\forall x \forall y \forall z\, (x + (y+z) = (x+y) + z)$

   8): $\forall x \forall y\, ((x \le y \wedge y \le x) \to x = y)$

3) a) In $\mathcal{L}_A$, find a Δ_0-formula which expresses the property that x is a square. Hence show that Q can capture the property of being a square.

   b) In $\mathcal{L}_A$, find a Δ_0-formula which expresses the property that x is *square-free*; that is, in its prime decomposition, no prime appears with a power higher than 1. Hence show that Q can capture the property of being square-free. You may assume that the property of being prime is expressible in $\mathcal{L}_A$ and capturable in Q.

4) Show that the sentence

$$\forall x\, (x \neq 0 \to \exists y\, (x = Sy))$$

   is derivable in PA. Hint: this would not be derivable in $\mathrm{I}\Delta_0$ if it were not already one of the axioms.

5) Show that the folllowing functions are primitive recursive. You should give the recursive definition explicitly, stating the functions g and h. You may use in your definitions any functions that have already been shown to be primitive recursive, either in the text or earlier in this problem.

   - $\min(x,y)$, $\max(x,y)$
   - $\mathrm{rm}(x,y)$, which is the function that returns the remainder when y is divided by x, with the convention that $\mathrm{rm}(0,y) = y$.
   - $f(x,y,z) = \sum_{z<y} g(x,z)$, where g is any primitive recursive function.