Math 3TP3 Truth and Provability Term 1 Autumn 2014–2015
Assignment 2 due 3 October 2014

(1) Give a totally different interpretation of the language of arithmetic, $\mathcal{L}_A$. You must state the universe of the interpretation, and the interpretation of the constant and function symbols. Give an example of a quantified sentence which is true in the standard interpretation and false in your interpretation.

(2) (thanks to P. Smith for this question) Let T be a theory in a language $\mathcal{L}$ which is computably axiomatisable, consistent and sufficiently strong. Let $\mathcal{L}'$ be a new language which contains $\mathcal{L}$, and let T' be a new theory in the language $\mathcal{L}'$ which contains T. Now let $T'|_{\mathcal{L}}$ be the theory which consists of all the theorems of T' that are expressed just in the language $\mathcal{L}$.

   (a) Give an example of such a situation; that is, state the languages $\mathcal{L}$ and $\mathcal{L}'$ and describe the theories T, T', $T'|_{\mathcal{L}}$.
   (b) Show that if $T'|_{\mathcal{L}}$ is consistent then it is sufficiently strong. revised
   (c) Show that if $T'|_{\mathcal{L}}$ is negation-complete then it is decidable.
   (d) Deduce that $T'|_{\mathcal{L}}$ is never negation-complete. Explain why, in the context of incompleteness theorems, this conclusion is of interest.

(3) Recall that part of the definition for a language to be sufficiently expressive is that there is a wff in the language which expresses a set which acts as the natural numbers under the interpretation. Give an example of a language expanding $\mathcal{L}_A$, an interpretation with universe $\mathbb{R}$ and a wff in the language whose set of realisations is either $\mathbb{N}$ or a set which acts like $\mathbb{N}$.

(4) Fill in the details of the proof of Theorem 10.4, that BA correctly evaluates any term.

(5) (a) In the game of Wff 'n Proof, prove that every wff can be decomposed into a principal connective followed by one or two shorter wffs.
   (b) Consider the restricted set of derivation rules Ko, Ki, Co, Ci, No, Ni and the derivation system that results from finitely many applications of the rules. Prove that each derivation rule is sound; that is, if ψ is deduced from φ by an application of one of the rules and v is any truth assignment such that $v(\varphi) = 0$, then also $v(\psi) = 0$.
   (c) Prove by induction that the derivation system is sound; that is, if $\varphi \vdash \psi$ and v is any truth assignment such that $v(\varphi) = 0$, then also $v(\psi) = 0$.