Math 3TP3 Truth and Provability Term 1 Autumn 2014–2015
Assignment 1

(1) Let $f : A \to B$ and $g : B \to C$ be functions.

(a) If f and g are both injective, prove that the composition defined by $g \circ f(x) = g(f(x))$ is also injective. Give examples to show that both of the hypotheses are necessary (that is, if either of the functions is not injective then it is possible that the composition is also not injective).

(b) If f and g are both surjective, prove that the composition is also surjective. Give examples to show that both of the hypotheses are necessary.

(2) Let A, and B be sets with characteristic functions c_A, c_B. Find the characteristic functions for $A \cup B$, $A \cap B$ and $\neg A$. Justify your answers.

(3) (thanks to P. Smith for this question) Recall that a set Σ is *computably enumerable* if it is the range of a computable function whose domain is the natural numbers.

(a) Define $f : \mathbb{N}^2 \to \mathbb{N}\setminus\{0\}$ by $f(m,n) = 2^m(2n+1)$. Prove that f is a computable bijection.

(b) Give explicit formulae (or at least, explicit algorithms) for the functions $fst : \mathbb{N}\setminus\{0\} \to \mathbb{N}$ and $snd : \mathbb{N}\setminus\{0\} \to \mathbb{N}$ which satisfy

$$fst(f(m,n)) = m \text{ and } snd(f(m,n)) = n.$$

(c) Use your functions fst and snd to construct a computable bijection $g : \mathbb{N}\setminus\{0\} \to \mathbb{N}^2$. Conclude that $\mathbb{N}^2$ is computably enumerable.

(4) Modify the game of WFF slightly, so that one is not allowed two sequential Ns. That is, an N can be put at the front of any wff provided it does not already begin with N. Find the length of the longest wff that can possibly be made when n dice are cast, for $n \in \{1,2,3,4,5,6\}$. Generalise to arbitrary n. (Notice that without the restriction on the Ns, the problem is very easy.)