Math 3TP3 Homework Set #2 Solutions

1) A different interpretation of $\mathcal{L}_A$:

Take the universe to be $\mathbb{R}$

S is interpreted as the function $x \mapsto e^x$

$+$ is interpreted as the function $(x,y) \mapsto x \cdot y$

$\cdot$ is interpreted as the function $(x,y) \mapsto x \cdot e^y$

$\bar{0}$ is interpreted as the element 1.

The sentence $\forall x\,(Sx = x + S\bar{0})$ is true in the standard interpretation. In the new interpretation, for any x

Sx is interpreted as e^x

$x + S\bar{0}$ is interpreted as $x \cdot e^1$ which is not e^x!

2) (a) ~~$\mathcal{L}_A$ is sufficiently strong (mentioned in Ch 8).~~

~~Take the theory T to be Q (which is finitely axiomatisable, hence computably axiomatisable)~~

~~Let $\mathcal{L}' = \mathcal{L}_A \cup \{c\}$, where c is a new constant symbol.~~

~~Let T' be the theory $Q \cup \{c \neq \bar{0}\}$.~~

~~T' can prove, for example, $\exists y\,(c = Sy)$, by Axiom III, the new axiom $c \neq \bar{0}$ and implication elimination. $T'|_{\mathcal{L}} = T$ – if we leave out~~

2) (a) Take T to be the theory Q in the language $\mathcal{L}_A$. By various remarks in the text, this is computably (in fact, finitely) axiomatisable, consistent and sufficiently strong.

Take $\mathcal{L}'$ to $\mathcal{L}_A \cup \{c\}$, where c is a new constant symbol. Take T' to be the theory $T \cup \{c \neq \bar{0}\}$. A sample theorem in T' is

$\exists y\,(c = Sy)$ using axiom IV, the new axiom $c \neq \bar{0}$ and implication elimination.

All new theorems in T' will involve c, so $T'|_{\mathcal{L}} = T$.

(b) A theory is sufficiently strong if it captures all effectively decidable numerical properties. Actually, the hypothesis of consistency is not needed. Since T is sufficiently strong, as $T \subseteq T'|_{\mathcal{L}}$, ~~also~~ $T'|_{\mathcal{L}}$ ~~captures all~~ proves everything that T does, so $T'|_{\mathcal{L}}$ is also sufficiently strong.

(c) By theorem 4.2, any consistent, computably axiomatised complete theory is decidable. Hence, if $T'|_L$ is complete then it is also decidable.

(d) By theorem 7.1, no consistent, sufficiently strong, computably axiomatised theory of arithmetic is decidable. If $T'|_L$ were complete, then by (c) it is decidable, which gives a contradiction. So $T'|_L$ is not complete.

We thus cannot hope to avoid the incompleteness theorem by increasing the language.

3) Take $\mathcal{L} = \mathcal{L}_A \cup \{f, c, R\}$

where f is a unary function symbol, c is a constant symbol and R is a binary relation symbol.

Define the interpretation to have universe $\mathbb{R}$, the usual interpretations of $S, +, \cdot, \bar{0}$, and

f is interpreted as the function $x \mapsto \sin(x)$

c — — — constant π

R — — — relation $x \leq y$.

$f(c \cdot x) = \bar{0} \,\&\, \bar{0} R x$

the formula ~~$\sin(\pi \cdot x) = \bar{0} \,\&\, \bar{0} R x$~~

expresses ~~defines~~ the natural numbers in $\mathbb{R}$.

~~Here is another example~~

4) Prove that BA correctly evaluates every term.

That is, if $val(\tau) = t$ then $BA \vdash \tau = \bar{t}$.

Proof is by induction on the construction of terms.

<u>Base case</u>: τ is a constant $\bar{0}$.

then $val(\tau) = 0$ and $BA \vdash \bar{0} = \bar{0}$, so $BA \vdash \tau = \bar{0}$.

<u>Successor case</u>: suppose τ is $S\sigma$, and that BA correctly evaluates σ.

Let $val(\sigma) = s$. By the assumption, $BA \vdash \sigma = \bar{s}$.

As S is a function, also $BA \vdash S\sigma = S\bar{s}$

i.e. $BA \vdash \tau = \overline{s+1}$

as $val(\sigma) = s$, $val(\tau) = val(S\sigma) = val(\sigma) + 1 = s+1$.

Thus $val(\tau) = s+1$ and $BA \vdash \tau = \overline{s+1}$, as required.

<u>addition case</u>: suppose τ is $\sigma_1 + \sigma_2$ and that BA correctly evaluates σ_1 and σ_2.

Let $val(\sigma_1) = s_1$, $val(\sigma_2) = s_2$. By the assumption,

$BA \vdash \sigma_1 = \bar{s}_1$ and $BA \vdash \sigma_2 = \bar{s}_2$.

As $+$ is a function, $BA \vdash \sigma_1 + \sigma_2 = \bar{s}_1 + \bar{s}_2$

By Theorem 10·2, as s_1, s_2 are numbers,

$BA \vdash \sigma_1 + \sigma_2 = \overline{s_1 + s_2}$

Now, $val(\tau) = val(\sigma_1 + \sigma_2) = s_1 + s_2$. So $BA \vdash \tau = \overline{s_1 + s_2}$.

Multiplication case: is exactly the same, using theorem 10.3.

5) (a) Proof is by induction on the construction of WFFs.

Base case ϕ is p.

Here we have the trivial connective of length 0.

Negation case ϕ is $N\psi$, ~~and~~ and assume result holds for ψ. Then ψ is shorter than ϕ, and the principal connective is N (actually, argument does not need the induction hypothesis.).

CAKE case ϕ is $Q\psi\theta$ where Q is any of C,A,K,E and ψ, θ are shorter WFFs. Then again, the Q is the principal connective.

(b) ~~Check~~ Check that each derivation rule preserves the truth assignment. Here are some of them

Ki $\dfrac{\phi \quad \psi}{\phi \,\&\, \psi}$ If $v(\phi) = 0$ and $v(\psi) = 0$ then $v(\phi \,\&\, \psi) = 0$ by the truth table for &.

Co $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ If $v(\varphi) = 0$ and $v(\varphi \to \psi) = 0$ then $v(\psi) = 0$ by the truth table for $\to$.

N: $\dfrac{\dfrac{\varphi}{\psi \quad \neg\psi}}{\neg\varphi}$ ~~the hypothesis is φ. Suppose $v(\varphi) = 0$. If~~

We have to show that, for every v st. $v(\varphi) = 0$, also $v(\neg\varphi) = 0$. That sounds like a problem. But the assumption is that $\varphi \vdash \psi$ and $\varphi \vdash \neg\psi$. At this point, we need to assume that these derivations are sound. Then we know that, for every v st. $v(\varphi) = 0$, also $v(\psi) = 0$ and $v(\neg\psi) = 0$. Since this is not possible, there is no v st. $v(\varphi) = 0$; that is, for every $v(\varphi) = 1$ or $v(\neg\varphi) = 0$, which is what we want.

(c) Proof is by induction on the length of the derivation of $\varphi \vdash \psi$. That is, we have a sequence $\varphi_0, \varphi_1, \ldots, \varphi_n$ with $\varphi_0 = \varphi$, $\varphi_n = \psi$ and each φ_i follows from the ones above by application of one of the rules in b).

Assume $v(\varphi_0) = 0$.

<u>n=1</u> The only derivation of length 1 is $\varphi_0 \vdash \varphi_0$, so $\psi = \varphi_0$ and $v(\psi) = 0$. (except for K_0, in which case φ_0 is $\theta_1 \wedge \theta_2$ and ψ is one of θ_1, θ_2).

<u>n+1</u> Assume result is true for a derivation of length at most n. The derivation of ψ comes from applying one of the derivation rules to φ_i, φ_j for $i, j \leq n$. Each of φ_i, φ_j is derived from φ_0 with a derivation of length $\leq n$, so by IH $v(\varphi_i) = v(\varphi_j) = 0$. By (b), application of the derivation rule preserves soundness, hence also $v(\varphi_{n+1}) = 0$, as required.