Math 3TP3 Homework # 1 due 19 September 2014

1) $f : A \to B$, $g : B \to C$

(a) f, g both injective, show $g \circ f$ is also injective.

Suppose $g \circ f(x_1) = g \circ f(x_2)$.

As $g \circ f(x_1) = g(f(x_1))$

$g \circ f(x_2) = g(f(x_2))$.

as g is injective, $f(x_1) = f(x_2)$.

as f is injective, $x_1 = x_2$.

thus $g \circ f$ is injective.

Consider $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$ $f(x) = x^2$ not injective

$g : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 1}$ $g(x) = e^x$ injective.

$$\left.\begin{aligned} g \circ f(-1) &= e^1 \\ g \circ f(1) &= e^1 \end{aligned}\right\} \text{ so } g \circ f \text{ is not injective}$$

Consider $f : \mathbb{R} \to \mathbb{R}$ $f(x) = x$ injective

$g : \mathbb{R} \to \mathbb{R}^{\geq 0}$ $g(x) = x^2$ not injective

$$\left.\begin{aligned} g \circ f(-1) &= 1 \\ g \circ f(1) &= 1 \end{aligned}\right\} \text{ so } g \circ f \text{ is not injective}$$

(b) f, g both surjective, show $g \circ f$ is also surjective.

Let $c \in C$. As g is surjective, there is $b \in B$ s.t. $g(b) = c$. As f is surjective, there is $a \in A$ s.t. $f(a) = b$. Then $g \circ f(a) = g(f(a)) = g(b) = c$.

Thus $g \circ f$ is surjective.

Consider $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$ $\quad$ f is not surjective onto $\mathbb{R}$.

$g : \mathbb{R} \to \mathbb{R}^{>0}$ $\quad f(x) = e^x$ $\quad$ g is surjective onto $\mathbb{R}^{>0}$.

$g \circ f$ is not surjective, as e^{-1} is not in the range.

Consider $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x$ $\quad$ surjective

$g : \mathbb{R} \to \mathbb{R}$ $\quad g(x) = e^x$ which is not surjective onto $\mathbb{R}$

$g \circ f$ is not surjective, as $-1 \cancel{= f(x)}$ is not in the range

2) $C_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$ $\qquad C_B(x) = \begin{cases} 0 & \text{if } x \in B \\ 1 & \text{if } x \notin B \end{cases}$

Let $C_{\neg A}(x) = 1 - C_A(x)$. If $x \notin A$ then $C_A(x) = 1$

so $C_{\neg A}(x) = 0$, as required

If $x \in A$ then $C_A(x) = 0$ so

$1 - C_A(x) = 1$ and $C_{\neg A}(x) = 1$, as required.

Let $C_{A \cup B}(x) = C_A(x)\, C_B(x)$.

If x is in either A or B, one of the factors is 0,

so the product is 0.

If x is in neither A nor B, both factors are 1,

so the product is 1.

Let $C_{A \cap B}(x) = C_A(x) + C_B(x) - C_A(x) \cdot C_B(x)$.

If $x \in A \cap B$ then everything is 0.

If x is in ~~one of~~ A and not in B then $C_A(x) = 0$,

so $C_A(x) \cdot C_B(x) = 0$, so $C_{A \cap B}(x) = 1$.

Similarly if x is in B and not A.

If x is in neither A nor B, then $C_{A \cap B}(x) = 1 + 1 - 1 = 1$.

3) (a) $f: \mathbb{N}^2 \to \mathbb{N}\setminus\{0\}$

$f(m,n) = 2^m(2n+1)$.

Suppose $f(m,n) = f(p,q)$, so $2^m(2n+1) = 2^p(2q+1)$

As $2n+1$ is odd and 2^m is only divisible by 2,

$2^m = 2^p$ and $2n+1 = 2q+1$.

Hence $m = p$ and $n = q$. Thus f is injective.

Let $p \in \mathbb{N}\setminus\{0\}$. There is a largest m s.t. $2^m \mid p$ (by unique factorisation of integers into primes). Given this m, $\frac{p}{2^m}$ is odd (possibly equal to 1).

Then $2 \mid \left(\frac{p}{2^m} - 1\right)$, so take $n = \left(\frac{p}{2^m} - 1\right)/2$.

Thus f is surjective.

(b) The above discussion shows how to calculate the inverse functions.

Given $p \in \mathbb{N}\setminus\{0\}$, let $fst(p) =$ greatest $m \geq 0$ st. $2^m \mid p$.

This is a computable function.

Then $snd(p) = \left(\frac{p}{2^m} - 1\right)/2$, which is also computable.

(e) Let $g : \mathbb{N} \setminus \{0\} \to \mathbb{N}^2$ be
$$g(p) = (\mathrm{fst}(p), \mathrm{snd}(p)).$$

g is surjective: for any $(m,n) \in \mathbb{N}^2$,
$$g(2^m(2n+1)) = (m,n).$$

g is injective: suppose $g(p) = g(q)$. Then $\mathrm{fst}(p) = \mathrm{fst}(q)$ and $\mathrm{snd}(p) = \mathrm{snd}(q)$.
Thus the highest powers of 2 dividing p and q are the same; that is $p = 2^m p'$, $q = 2^m q'$.
As $\mathrm{snd}(p) = \mathrm{snd}(q)$, $p' = q'$. So $p = q$.

g is computable because fst and snd are computable.

Technically, to conclude that $\mathbb{N}^2$ is computably enumerable, we need a computable function $h : \mathbb{N} \to \mathbb{N}^2$. So define $h(n) = g(n+1)$.

4) First note that, if two or more sequential N's are allowed, we can get a WFF of any length $n > 1$ by taking $\underbrace{N \ldots N}_{n-1} p$.

There is a WFF of length n for any $n \in \{1, 2 \ldots 6\}$.

p	1
Np	2
Apq	3
$ANpq$	4
$ANpNq$	5
$EApqApq$	6.

Claim for any $n \in \mathbb{N}$, there is a WFF of length n.

Pf by induction on n. The base case is provided above. So assume there is a WFF φ of length n.

Case 1 φ does not begin with N. Then $N\varphi$ is a WFF of length $n+1$.

Case 2 φ begins with N. Let φ' be φ with the leading N removed (a WFF, by construction of WFFs), of length $n-1$). Then $A\varphi' p$ is a WFF of length $n+1$, as required.