Math 2R 07-08 Solutions to Practice Problems # 2

1) Look them up!

2) $T: V \to V$ linear operator, u, v eigenvectors of T; say $T(u) = \lambda_1 u$, $T(v) = \lambda_2 v$, $\lambda_1 \neq \lambda_2$.

Suppose there are $a, b \in \mathbb{R}$ s.t.

$$au + bv = 0. \qquad (1)$$

Apply T:

$$T(au+bv) = T(0)$$

$$aT(u) + bT(v) = 0$$

$$a\lambda_1 u + b\lambda_2 v = 0 \qquad (2)$$

Solve for v in eqns (1) and (2):

$(2) - \lambda_1(1)$: $\quad a\lambda_1 u + b\lambda_2 v - (a\lambda_1 u + b\lambda_1 v) = 0$

$$b(\lambda_2 - \lambda_1)\, v = 0$$

as ~~$b \neq 0$~~ $\bar{v} \neq 0$ and $\lambda_2 \neq \lambda_1$, must have $b = 0$.

Hence $a = 0$ also.

thus u, v are independent

3) $T: \mathbb{P}_2 \to \mathbb{P}_2$

$$T(p(x)) = p(x) + p(-x).$$

a $T(a+bx+cx^2) = a+bx+cx^2 + a+b(-x)+c(-x)^2$

$$= 2a + 2cx^2$$

$\ker(T) = \{ p(x) : T(p(x)) = 0 \}$

$= \{ a+bx+cx^2 : \quad 2a+2cx^2 = 0+0x+0x^2 \}$

$= \{ a+bx+cx^2 : \quad a=0 \text{ and } c=0 \}$

thus $\ker(T) = \text{span}\{x\}$.

Since $\dim(\ker(T)) = 1$ and $\dim(\mathbb{P}_2) = 3$, it follows by the dimension theorem that $\dim(\text{im}(T)) = 2$.

$\text{im}(T) = \{ q(x) = q_0 + q_1 x + q_2 x^2 \mid$ there exists $p(x)$ with $q(x) = T(p(x)) \}$

$= \{ q_0 + q_1 x + q_2 x^2 :$ there exist $a, b, c \in \mathbb{R}$ with $q(x) = 2a + 2cx^2 \}$

since we can solve $\frac{1}{2} q_0 = a$, $c = \frac{1}{2} q_2$, $b = 0$

$\text{im}(T) = \{ q_0 + q_2 x^2 \} = \text{span}\{1, x^2\}$.

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4) $B = \{1,\ x-2,\ x^2+3\} = \{b_1, b_2, b_3\}$

$$P_{B\leftarrow E} = \left(c_B(e_1) \quad c_B(e_2) \quad c_B(e_3) \right)$$

$e_1 = 1 = b_1$

$e_2 = x = x - 2 + 2 = 1\cdot b_2 + 2b_1$

$e_3 = x^2 = x^2 + 3 - 3 = 1\cdot b_3 - 3b_1$

$$P_{B\leftarrow E} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$p(x) = a + bx + cx^2$

$$c_E(p(x)) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad c_B(p(x)) = P_{B\leftarrow E}\, c_E(p(x))$$

$$= \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} a + 2b - 3c \\ b \\ c \end{pmatrix}$$

$$p(x) = (a + 2b - 3c)b_1 + b\,b_2 + c\,b_3$$

$$= (a + 2b - 3c)\,1 + b(x-2) + c(x^2+3).$$

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5) $M_{EB} = \left(c_E T(b_1) \quad c_E T(b_2) \quad c_E T(b_3) \right)$

$T(b_1) = T(1) = 2$

$T(b_2) = T(x-2) = x-2 + (-x)-2 = -4$

$T(b_3) = T(x^2+3) = x^2+3 + (-x)^2+3 = 2x^2+6$

$$M_{EB} = \begin{pmatrix} 2 & -4 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

6) $$A = \begin{pmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{pmatrix}$$

Solve $\det(yI - A) = 0$; get $(y+1)(y-2)^2 = 0$

eigenvalues are -1 with multiplicity 1
2 with multiplicity 2.

Solve $(-I-A)X = 0$: get $X = t\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

$$G_{-1} = \text{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

Solve $(2I-A)X = 0$: get $X = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

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Solve $(2I-A)^2 X = 0$: get $t\begin{pmatrix}1\\1\\1\end{pmatrix}$ or $s\begin{pmatrix}0\\1\\-1\end{pmatrix}$.

$$G_2 = \text{span}\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\}$$

~~Ete~~ By calculation, $A\begin{pmatrix}1\\3\\0\end{pmatrix} = -1\begin{pmatrix}1\\3\\0\end{pmatrix}$, hence $AG_{-1} = G_{-1}$.

So G_{-1} is invariant.

$A\begin{pmatrix}1\\1\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\\1\end{pmatrix} \in G_2$ $\quad A\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\3\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + 2\begin{pmatrix}0\\1\\-1\end{pmatrix}$

$\cancel{A\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}-9\\3\\-1\end{pmatrix} = -9\begin{pmatrix}1\\1\\1\end{pmatrix}}$ Thus $AG_2 \subseteq G_2$.

Basis of $\mathbb{R}^3$ is $B = \left\{ \begin{pmatrix}1\\3\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \right\}$

Block diagonal form of A is $\begin{pmatrix}-1 & 0 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{pmatrix}$.

transition matrix is $P = P_{\cancel{B \leftarrow E}\ E \leftarrow B} = \left(c_E(b_1)\ \ c_E(b_2)\ \ c_E(b_3) \right)$

$$= \begin{pmatrix}1 & 1 & 0\\ 3 & 1 & 1\\ 0 & 1 & -1\end{pmatrix}$$