Solutions to: Practice Problems for Midterm 1.

1) Look it up!

2) True: $u \in \operatorname{span}\{v_1, \dots, v_n\}$

$$\Rightarrow u = a_1 v_1 + \dots + a_n v_n \text{ for scalars } a_1, \dots, a_n$$

$$\Rightarrow a_1 v_1 + \dots + a_n v_n - u = 0.$$

This is a dependence relation with at least one non-zero coefficient. (which one?)

3) Correction: show $(-1)\bar{v} = -\bar{v}$ for all vectors $\bar{v}$.

By definition (axiom A4) $-\bar{v}$ is the vector in V with the property that $-\bar{v} + \bar{v} = \bar{0}$.

Now $(-1)\bar{v} + \bar{v} = (-1+1)\bar{v}$ S3

$= 0\bar{v}$

If $0\bar{v} = \bar{0}$ then we are done, for then $(-1)\bar{v}$ satisfies the defining property of $-\bar{v}$.

Now, $0\bar{v} = (0+0)\bar{v}$ as $0+0=0$

$= 0\bar{v} + 0\bar{v}$ S3

Hence $\bar{0} = 0\bar{v}$ by cancellation, as required.

4) X and Y are two sets which both span the same vector space. This does not imply that they have a common element.

For example, $X = \{(1,0), (0,1)\}$ and

$$Y = \{(1,1), (1,-1)\}$$

both span $\mathbb{R}^2$, but $X \cap Y = \phi$.

5) The set $\{(0,0,3,2), (0,1,7,4)\} = B$ is clearly independent.

Also, clearly $(1,0,0,0) \notin \text{span}(B)$, so

$$B_1 = \{(0,0,3,2), (0,1,7,4), (1,0,0,0)\} \text{ is independent.}$$

Since $\dim(\mathbb{R}^4) = 4$, need one more vector to get a basis.

Any vector in the span of B_1 has the form

$(a, b, 3a+7b, 2a+4b)$. To get a vector not in the span of B_1, take ~~a=~~ $b = c = 0$ and then choose the third and fourth coordinates (c_3, c_4) so that there is no solution to $3a = c_3$, $2a = c_4$.

Say $c_3 = 1$, $c_4 = 5$.

Thus $\{(0,0,3,2), (0,1,7,4), (1,0,0,0), (0,0,1,5)\}$

is a basis for $\mathbb{R}^4$ containing the original set.

6) $U = \{ p(x) \in \mathbb{P}_4 : p(x) = p(-x) \text{ for all } x \in \mathbb{R}\}$.

Show U is a subspace:

$\bar{0}(x) = 0 = \bar{0}(-x)$, so $\bar{0} \in U$.

Suppose $p, q \in U$. Then ~~p(x)~~ $(p+q)(x) = p(x) + q(x)$

$= p(-x) + q(-x)$ by assumption

$= (p+q)(-x)$.

Hence $p+q \in U$.

Suppose $a \in \mathbb{R}$. Then

$(ap)(x) = a\,p(x) = a\,p(-x)$ as $p \in U$

$= (ap)(-x)$.

Thus $ap \in U$.

Hence U is a subspace.

U is clearly a proper subspace of $\mathbb{P}_4$, hence

$\dim(U) < \dim(\mathbb{P}_4) = 5$.

Let $p_0(x) = 1$, $p_2(x) = x^2$, $p_3(x) = x^4$.

Clearly $p_0, p_1, p_2 \in U$. Also $\{p_0, p_1, p_2\}$ is independent as the p_i have different degrees.

Thus $\dim(U) \geq 3$.

If $U \neq \text{span}\{p_0, p_1, p_2\}$ then U must contain a polynomial with a term of odd degree; say $q(x) = p(x) + ax^3 + bx$ where $p(x) \in \text{span}\{p_0, p_1, p_2\}$.

But
$$q(-x) = p(-x) + a(-x)^3 + b(-x)$$
$$= p(x) + -ax^3 - bx \neq p(x) + ax^3 + bx$$
unless $a = b = 0$.

Thus $\text{span}\{p_0, p_1, p_2\} = U$.

7) $V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{22} : a + c = b + d \right\}$

Since $0 + 0 = 0 + 0$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in V$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in V$.

then $A + A' = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}$

$$\begin{aligned}(a+a') + (c+c') &= (a+c) + (a'+c') \\ &= (b+d) + (b'+d') \quad \text{as } A, A' \in V \\ &= (b+b') + (d+d'). \quad \text{Thus } A + A' \in V.\end{aligned}$$

Also, $rA = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$

$$\begin{aligned} ra + rc = r(a+c) &= r(b+d), \quad \text{as } A \in V \\ &= rb + rd. \end{aligned}$$

Thus $rA \in V$.

Clearly, $V \neq M_{22}$, so $\dim(V) < 4$.

$\left.\begin{array}{l} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in V \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in V \end{array}\right\}$ clearly these two are independent.

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in V$ This is not in the span of the first two – look at the $(2,1)$ coefficient.

Hence these three vectors together are independent, and thus a basis for V.

8) Find four independent ~~vectors~~ matrices satisfying $A^2 = A$.

One method is to guess, but here is a more organized procedure.

Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A^2 = A$ implies that

$$a^2 + bc = a \quad (1) \qquad (a+d)b = b \quad (2)$$

$$c(a+d) = c \quad (3) \qquad bc + d^2 = d \quad (4).$$

There are infinitely many solutions to these equations; we need to find four independent ones.

$(2) \Rightarrow a+d = 1$ or $b = 0$

$(3) \Rightarrow a+d = 1$ or $c = 0$.

Take $b = c = 0$: then $(1) \Rightarrow a^2 = a \Rightarrow a = 0$ or 1

$(4) \Rightarrow d^2 = d \Rightarrow d = 0$ or 1.

Take $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ – clearly independent.

Now take one of b, c to be 0. Then still get $a = 0$ or 1, $d = 0$ or 1.

So take $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Easy to check that these four matrices are independent.