Math 2R03, Project 2 Due Wednesday, November 28 2007, IN CLASS

Differential equations and Block Diagonal Form

Recall from the last project that we showed that a second-order linear differential equation can be written as a system of two first-order simultaneous linear differential equations, and that the solutions to this system are given by the exponential of a matrix. In the same way, any first-order linear differential equation involving n functions can be written in matrix form, and the solution will be given in exponential form. This time, we will discover how the block diagonal form of the matrix can be used to understand the solutions.

Consider the system of differential equations

$$\begin{aligned} x'(t) &= x(t) + 4y(t) \\ y'(t) &= -2x(t) + 5y(t) \\ z'(t) &= -2x(t) + 4y(t) + z(t). \end{aligned}$$

1) Write this equation in the form $X'(t) = AX$, where $X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$.

The unique solution to this system of differential equations is $X(t) = e^{tA}X_0$, where $X(0) = \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}$ is the initial condition. The set of solutions to this differential equation is a vector space of dimension 3. Recall that, by definition,

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

In general, A^n is difficult to calculate, but we can make it easier by converting A to block diagonal form.

2) Find a basis B for $\mathbf{R}^3$ so that

$$M = M_B(A) = \begin{pmatrix} 3 & 2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the change of basis matrix P so that $A = P^{-1}MP$.

Since $e^{tA} = P^{-1}\sum_{n=0}^{\infty}\left(\frac{t^n}{n!}M^n\right)P = P^{-1}e^{tM}P$, we just need to calculate M^n for all n. Write $M = 3J + 2K + L$, where $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $K = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

and $L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is not in general true that e^{A+B} is the same as $e^A e^B$. However, J, K and L have block diagonal form. Because of this, J and K both commute with L. Furthermore, J commutes with K.

3) Use all the above information to show that

$$e^{tM} = e^{t(3J+2K+L)} = e^{t3J} e^{t2K} e^{L}.$$

A bit more work, similar to that which you did on the last project, reveals that

$$e^{t2K} = \begin{pmatrix} \cos(2t) & \sin(2t) & 0 \\ -\sin(2t) & \cos(2t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and hence that

$$e^{tM} = \begin{pmatrix} e^{3t}\cos(2t) & e^{3t}\sin(3t) & 0 \\ -e^{3t}\sin(3t) & e^{3t}\cos(2t) & 0 \\ 0 & 0 & e^{t} \end{pmatrix}.$$

A basis for the space of solutions to the differential equation $X' = MX$ is $\{F_1, F_2, F_3\}$, where $F_1 = e^{tM}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $F_2 = e^{tM}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $F_3 = e^{tM}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

4) Describe each function $F_i(t)$, for $t \in [0, \infty)$. Hence describe the linear combinations $s_1F_1(t) + s_2F_2(t)$, $s_1F_1(t) + s_3F_3(t)$, $s_2F_2(t) + s_3F_3(t)$, $s_1F_1(t) + s_2F_2(t) + s_3F_3(t)$. Be sure to consider both s_3 positive and s_3 negative. Notice that the vector $\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$ gives the initial conditions of the system of differential equations.

5) Use your answer to 4) and the change of basis matrix P to describe the solutions to the original differential equation $X' = AX$.