Math 2R03, Project #1 Due Friday, October 11 2007, IN CLASS

Linear algebra and differential equations

Recall from your calculus course that the solution to the differential equation $f'(t) = af(t)$ is the family of functions $f(t) = Ce^{ta}$, where C is any real number (and in fact, $C = f(0)$). From the point of view of linear algebra, we say that

$$U = \{f(t) : f'(t) = af(t)\}$$

is a subspace of the space of all differentiable functions, and $\dim(\mathrm{U}) = 1$. The basis is $\{e^{ta}\}$ and a proof that every other element of U is a linear combination of this basis element is as follows: consider $g \in U$. We want to show that $g(t) = re^{ta}$, for some scalar r. Equivalently, we want to show that $e^{-ta}g(t) = r$, or equivalently that $e^{-ta}g(t)$ is a constant function. But this follows immediately, because it is easy to show that

$$\frac{d}{dt}\big(e^{-ta}g(t)\big) = 0.$$

We want to apply the same ideas to find solutions to higher order differential equations. Consider the differential equation

$$f''(t) = af'(t) + bf(t).$$

1) Let $U = \{f : f''(t) = af'(t) + bf(t)\}$. Prove that U is a subspace of the space of differentiable functions.

We use the following trick to convert the second-order differential equation to a pair of first-order simultaneous equations. Write $f'(t) = g(t)$ and then $g'(t) = ag(t) + bf(t)$. This can now be written as the matrix equation

$$\begin{pmatrix} f'(t) \\ g'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix} \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \text{ or } \vec{F}'(t) = A\vec{F}(t). \tag{1}$$

We would like to say that the solution to this first-order matrix differential equation is given by an exponential function, but first we need to understand what this means.

Recall that the function e^t converges to its Taylor series for all values of t; so we can write

$$e^{ta} = \sum_{n=0}^{\infty} \frac{1}{n!}(ta)^n.$$

It turns out that we can use this infinite series to define the exponential of a matrix (which will itself be a matrix). We define

$$e^{tA} = \sum_{n=0}^{\infty} \frac{1}{n!}(tA)^n,$$

where A is any square matrix, t is a scalar. This function can be shown to be well-defined and satisfy all the usual rules for exponentials, including that

$$\frac{d}{dt}\left(e^{tA}\right) = Ae^{tA}.$$

2) As outlined above, prove that $\vec{F}(t) = e^{At}\vec{F}_0$ gives all solutions to the equation (1). Deduce that $\dim(\mathrm{U}) = 2$.

A classical example of physical system modelled by this differential equation is that of a mass suspended on a spring, moving through a viscous medium. Consider the differential equation as above, with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & v \end{pmatrix}; \qquad \vec{F}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

3) Describe, using words and graphs, the solution to the differential equation in each of the three cases

i) $v = 0$;
ii) $0 < v < 2$;
iii) $2 < v$.

(You may choose representative values of v in cases ii) and iii) if you find this easier). You may use any further piece of theory, or computer software that you wish, but you must explain fully how you get your conclusions from the solutions given in 2) ("I found a book that says the solutions have this form" is not good enough).