Name: *Student number:*

1. (10) Each part of the following question is worth 2 points. NO PARTIAL CREDIT will be given.

(i) Let V be a vector space, B and D two ordered bases of V, $T : V \to V$ a linear operator. State the relationship between $M_B(T)$ and $M_D(T)$.

$$M_B(T) = P_{D\leftarrow B}^{-1} M_D(T) P_{D\leftarrow B}$$

or $M_B(T)$ and $M_D(T)$ are similar matrices

(ii) Define the term *linear transformation.*

A function $T : V \to W$ is a linear transformation if for all $u, v \in V$, $a \in \mathbb{R}$ $T(u+v) = T(u) + T(v)$ and $T(au) = aT(u)$.

(iii) Define the term *one-to-one* for a linear transformation.

the linear transformation T is one-to-one if $T(v) = T(u) \Rightarrow v = u$.

(iv) Define the term *T-invariant subspace.*

A subspace U of the vector space V is T-invariant if for all $u \in U$, $T(u) \in U$.

(v) Let $T : \mathbf{P}_3 \to \mathbf{R}$ be the linear transformation defined by $T(a + bx + cx^2) = a$. Find $\ker(T)$.

$$\begin{aligned}\ker(T) &= \{ a+bx+cx^2 : T(a+bx+cx^2) = 0\} \\ &= \{ a+bx+cx^2 : a = 0\} \\ &= \{ bx + cx^2\} \\ &= \text{span}\{x, x^2\}\end{aligned}$$

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2. (5)

(i) Let $T : \mathbf{P}_2 \to \mathbf{P}_2$ be defined by $T(x+1) = x$, $T(x-1) = 1$, $T(x^2) = 0$. Find $T(2+3x-x^2)$.

$$2 + 3x - x^2 = \frac{5}{2}(x+1) + \frac{1}{2}(x-1) - 1x^2$$

so $$T(2+3x-x^2) = T\left(\frac{5}{2}(x+1) + \frac{1}{2}(x-1) - 1x^2\right)$$

$$= \frac{5}{2}T(x+1) + \frac{1}{2}T(x-1) - 1\,T(x^2)$$

$$= \frac{5}{2}x + \frac{1}{2}\cdot 1 - 0.$$

(ii) Let $T : V \to W$, $S : W \to U$ be linear transformations. Suppose that T and S are both onto. Show that ST is also onto.

Let $u \in U$. As S is onto U, there is $w \in W$ with $S(w) = u$.
As T is onto W, there is $v \in V$ with $T(v) = w$.

Hence $ST(v) = S(T(v)) = S(w) = u$.
Thus $u \in \mathrm{im}(ST)$ for any $u \in U$.
That is $U \subseteq \mathrm{im}(ST)$, as required.

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3. (5) Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation defined by

$$T(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} a+2b \\ -a \\ a \end{pmatrix}.$$

Find $\ker(T)$. Use the dimension theorem to calculate $\dim(\mathrm{im}(T))$. Then find a basis for $\mathrm{im}(T)$.

$$\ker(T) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : T\begin{pmatrix} a \\ b \end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : \begin{pmatrix} a+2b \\ -a \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

$$\dim(\mathbb{R}^2) = \dim(\ker(T)) + \dim(\mathrm{im}(T))$$

$$2 = 0 + \dim(\mathrm{im}(T)).$$

$$\mathrm{im}(T) = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} : w = a,\ v = -a,\ u = a+2b \text{ for some } a, b \in \mathbb{R} \right\}$$

basis is $\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ as these vectors are clearly independent, and there are two of them.

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4. (7) Let $T : \mathbf{P}_2 \to \mathbf{P}_3$ be the linear transformation defined by

$$T(p(x)) = xp(x-3).$$

For example, $T(x^2-1) = x((x-3)^2-1) = x^3-6x^2+8x$. Consider the following two bases for $\mathbf{P}_2$ and $\mathbf{P}_3$ respectively: $B = \{1, x, x^2\}$, $D = \{1, x-3, (x-3)^2, (x-3)^3\}$. Find $M_{DB}(T)$. Use $M_{DB}(T)$ to calculate $T(a+bx+cx^2)$ expressed in the basis D. (No credit will be given for calculating T by another method, although you may use this to check your answer.)

$$M_{DB}(T) = \left(c_D(T(b_1)), \ c_D(T(b_2)) \ \ c_D(T(b_3)) \right)$$

$$T(b_1) = T(1) = x = 1.(x-3) + 3.1$$

$$T(b_2) = T(x) = x(x-3) = 1(x-3)^2 + 3(x-3)$$

$$T(b_3) = T(x^2) = x(x-3)^2 = 1.(x-3)^3 + 3(x-3)^2$$

$$M_{DB}(T) = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T(a+bx+cx^2) = c_D^{-1}\left(M_{DB}\, c_B(a+bx+cx^2)\right)$$

$$= c_D^{-1} \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= c_D^{-1} \begin{pmatrix} 3a \\ a+3b \\ b+3c \\ c \end{pmatrix} = 3a + (a+3b)(x-3) + (b+3c)(x-3)^2 + c(x-3)^3$$

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5. (6) Consider the following basis for M_{22}:

$$B = \{\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 4 & -1 \end{pmatrix}\}.$$

Find the change-of-basis matrix $P_{B\leftarrow E}$, where E is the standard basis. Use $P_{B\leftarrow E}$ to write $A = \begin{pmatrix} 4 & 2 \\ -4 & 12 \end{pmatrix}$ as a linear combination of the basis vectors in B.

$$P_{B\leftarrow E} = \left(c_B(e_1) \quad c_B(e_2) \quad c_B(e_3) \quad c_B(e_4) \right)$$

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{8}{5}\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix} - \frac{3}{5}\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} - \frac{4}{5}\begin{pmatrix} 0 & 0 \\ 4 & -1 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{4}{5}\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix} - \frac{4}{5}\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 0 & 0 \\ 4 & -1 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -\frac{3}{10}\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix} + \frac{3}{10}\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 0 & 0 \\ 4 & -1 \end{pmatrix}$$

$$e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{6}{5}\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} + \frac{4}{5}\begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix} + \frac{6}{5}\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 0 & 0 \\ 4 & -1 \end{pmatrix}$$

$$P_{B\leftarrow E} = \begin{pmatrix} 8/5 & 4/5 & -3/10 & -6/5 \\ -2/5 & -1/5 & 1/5 & 4/5 \\ -3/5 & -4/5 & 3/10 & 6/5 \\ -4/5 & -2/5 & 2/5 & 3/5 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 \\ -4 & 12 \end{pmatrix} = C_B^{-1} P_{B\leftarrow E} \begin{pmatrix} 4 \\ 2 \\ -4 \\ 12 \end{pmatrix} = C_B^{-1} \begin{pmatrix} -26/5 \\ 34/5 \\ 46/5 \\ 8/5 \end{pmatrix}$$

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$$= -\frac{26}{5}\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} + \frac{34}{5}\begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix} + \frac{46}{5}\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} + \frac{8}{5}\begin{pmatrix} 0 & 0 \\ 4 & -1 \end{pmatrix}$$

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6. (7) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation defined by

$$T\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} a+c \\ 2b \\ 2c \end{pmatrix}.$$

Find two T-invariant subspaces U_1, U_2 such that $\mathbf{R}^3 = U_1 \oplus U_2$. Give a basis $B = B_1 \cup B_2$ of $\mathbf{R}^3$, where B_1 is a basis of U_1, B_2 is a basis of U_2. Find $M_B(T)$ (which should be in block diagonal form).

Let $U_1 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$ $\quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in U_1,$

so U_1 is T-invariant.

Let $U_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$ $\quad T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in U_2$

$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in U_2$

so U_2 is T-invariant.

$B = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$

$$M_B(T) = \left(c_B(T(b_1)) \quad c_B(T(b_2)) \quad c_B(T(b_3)) \right)$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$