Lecture 9: Exs of finding bases.

V vectorspace of dim n.

B subset with n elts.

Claim: B is independent if and only if B spans V.

Pf. First, suppose B is independent.

To show $\operatorname{span}(B) = V$, suppose not.

By Going Up, B can be extended to a basis, B_1. But then B_1 has more than n elements. This contradicts $\dim(V) = n$.

For the other direction, suppose B is a spanning set. If B is not independent, then by Going Down, can cut down to a basis, say B_1.

But then B_1 has fewer than n elts; again contradicts $\dim(V) = n$.

Ex of Going Up:

Consider $B = \{x^2, x-1, 2x\}$. Expand B to a basis of $\mathbb{P}_4$.

$\dim(\mathbb{P}_4) = 5$.

Verify that B is independent.

Suppose $r_1 x^2 + r_2(x-1) + r_3(2x) = 0$

Compare coefficients of powers of x:

$$\left.\begin{array}{lll} x^2: & r_1 & = 0 \\ x: & r_2 + 2r_3 & = 0 \\ x^0: & -r_2 & = 0 \end{array}\right\} \Longrightarrow r_1 = r_2 = r_3 = 0$$

Hence B is independent.

$x^3 \notin \text{span}(B)$, so by the proof of the Going Up Lemma,

$B \cup \{x^3\} = \{x^2, x-1, 2x, x^3\} = B_1$ is independent.

$x^4 \notin \text{span}(B_1)$, so

$B_2 = \{x^2, x-1, 2x, x^3, x^4\}$ is independent.

Since B_2 has 5 elements, and $\dim(\mathbb{P}_4) = 5$

B_2 is a basis for $\mathbb{P}_4$, which includes B, as required.

Ex. Consider $B = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$.

Expand B to a basis of the space of symmetric 2×2 matrices.

Consider $v \in \text{span}(B)$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = r_1 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + r_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} r_1 & r_1 + r_2 \\ r_1 + r_2 & r_2 \end{pmatrix}$$

$a = r_1$ $\qquad$ $c = r_1 + r_2 \Rightarrow b = c$.

$b = r_1 + r_2$ $\qquad$ $d = r_2$

~~Th~~ Span $(B) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{22} : b = c = a + d \right\}$

Thus $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \notin \text{Span}(B)$, and is symmetric.

So $\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right\}$ is indt, hence spans the symmetric 2×2 matrices.

Lemma: Let W be a subspace of V, and suppose $\dim(W) = \dim(V) = n$. Then $W = V$.

<u>Pf</u>. Any basis for W has n elts. By our statement at the beginning of class, this basis spans V. Thus $V \subseteq W$. Thus $V = W$.

<u>Ex</u>. S = symmetric 2×2 matrices

S is a subspace of M_{22}, and it is a proper subspace of M_{22}. Thus by the lemma, $\dim(S) < \dim(M_{22}) = 4$.

Ex of Going Down:

Consider $S = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \right\}$

Find span(S), and a ~~basis~~ subset of S which is a basis for span(S).

Clearly $S_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \right\}$ is ind t.

Is $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in \operatorname{Span}(S_1)$?

i.e. can I solve $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \overset{?}{=} r_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + r_2 \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$

$0 = r_1 - r_2 \qquad 0 = 0r_1 + 2r_2$

$\Rightarrow r_1 = r_2 = 0 \qquad$ No.

Thus $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ is ind t.

Same method $\{ \underline{\text{4 matrices}} \}$ also ind t

Hence $\dim(\operatorname{span}(S)) = 4$, and $\operatorname{span}(S) = M_{22}$.

Title: Sep 26 - 1:11 PM (7 of 7)