Midterm 1

Scheduled conflicts: send email haskell@math.mcmaster.ca

Wednesday 3 October

7:00 – 8:30 pm

Material: sections 6.1 — 6.4 of text book

(Everything up to Friday this week.)

Lecture 8: more on bases of vector spaces

Recall: Fund Thm Suppose vector space V can be spanned by n vectors. Then any ind^t set in V has at most n vectors.

<u>Corollary</u> Let V be a vector space of dimension n. Then:

1) any set in V with more than n vectors is dependent.
2) any set of fewer than n vectors does not span V.
3) Going up: Any independent set of vectors in V can be expanded to a basis.

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4) Going down: any spanning set for V can be cut down to a basis.

5) If B is a set with n vectors, then B is independent if and only B spans V.

<u>Proof</u>.

1) ~~By the Fund Thm~~, Since V has dimension n, it has a spanning set with n elts. By Fund Thm, no ind^t set has more than n elts.

2) Suppose a set with $m<n$ elts spans V. By the Fund Thm, no ind^t set has more than m elts. But $\dim(V)=n$, so V has a basis with $n>m$ elts. Contradiction. Hence no set with fewer than n elements spans V.

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3) Let $B = \{v_1, \dots, v_m\}$ an ind^t set in V, $m \leq n$. Suppose span$(B) \neq V$.

Let $v_{m+1} \in V \setminus \text{span}(B)$. Then

$B_1 = \{v_1, \dots, v_m, v_{m+1}\}$ is independent

(for otherwise $v_{m+1} \in \text{span}(B)$).

If span$(B_1) \neq V$ then repeat.

Claim: $B_{n-m} = \{v_1, \dots, v_m, v_{m+1}, \dots v_n\}$ spans V.

<u>Pf</u> If not, there is $w \in V$, $w \notin \text{Span}(B_{n-m})$

Then $\{v_1, \dots, v_m, v_{m+1}, \dots, v_n, w\}$ is independent. By i), this is a contradiction.

Hence B_{n-m} is a basis for V, containing B.

4) Let $S=\{w_1,\dots,w_k\}$ be a spanning set, $k \geq n$. If S is not independent, then one of its elements is a linear combination of the others.

Relabelling, WMA $w_k \in \mathrm{span}\{w_1,\dots,w_{k-1}\}$

Let $S_1 = \{w_1,\dots,w_{k-1}\}$; then $V = \mathrm{span}(S_1)$.

Why? Let $v \in V$. Since $V = \mathrm{span}(S)$,

$$v = r_1w_1 + r_2w_2 + \dots + r_kw_k$$

We know $w_k = a_1w_1 + \dots + a_{k-1}w_{k-1}$

Substitute:

$$v = r_1w_1 + \dots + r_{k-1}w_{k-1} + r_k(a_1w_1 + \dots + a_{k-1}w_{k-1})$$

Using commutativity of vector addition, distributivity of scalar mult^n,

$$v = (r_1 + r_ka_1)w_1 + \dots + (r_{k-1} + r_ka_{k-1})w_{k-1}$$

Thus $v \in \mathrm{span}(S_1)$.

Now continue: if S_1 is not ind^t, one of its elements, say $w_{k-1} \in \mathrm{span}\{w_1,..w_{k-2}\}$

Repeat.

<u>Claim</u>: S_{k-n} is independent. If not, then we could remove one elt as above, and S_{k-n+1} would still be a spanning set. But this contradicts 2).