Lecture 7: Calculating Dimension

<u>Defn</u> The <u>dimension</u> of a vector space is the number of vectors in any linearly independent, spanning set, if that number is finite. If there is no finite basis we say the vector space is <u>infinite-dimensional</u>.

$\underline{\mathbb{R}^n}$ Standard basis is

$$\left\{ \underset{e_1}{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \ldots \underset{e_n}{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}} \right\}$$

Clearly $\{e_1 \ldots e_n\}$ is independent.

If $r_1 e_1 + \ldots + r_n e_n = \bar{0}$, then

$r_1 = r_2 = \ldots = r_n = 0$, as

each r_i appears in only one of

the n simultaneous eqns.

Clearly, $\{e_1, \ldots e_n\}$ spans $\mathbb{R}^n$:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \ldots + a_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Hence dimension of $\mathbb{R}^n$ is n.

$\underline{M_{22}}$ standard basis is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$E_1 \quad E_2 \quad E_3 \quad E_4$

This set is clearly lin. ind. and spans M_{22}.

dimension of M_{22} is 4

M_{mn} std basis is $\{ E_{ij} \in M_{mn}$: E_{ij} has 1 as (i,j) coefficient, 0's elsewhere $\}$

$1 \le i \le m$
$1 \le j \le n$

This set has $m \times n$ elements, so $\dim(M_{mn}) = mn$.

$\mathbb{P}_n$

$\{1, x, x^2, x^3, \ldots, x^n\}$

Clearly a spanning set.

Clearly lin. indep.

$$\dim(\mathbb{P}_n) = n+1.$$

Remark any set of polynomials of distinct degree is independent.

Pf. Let $p_1, p_2, \ldots, p_n$ be the polynomials, $\deg(p_i) = d_i$. Assume that d_i are all different. WMA $d_1 < d_2 < \ldots < d_n$.

Suppose $r_1 p_1 + r_2 p_2 + \ldots + r_n p_n = \bar{0}$ (1)

Write $p_i(x) = a_i x^{d_i} +$ lower terms

Consider coefficient of x^{d_n} in eqn (1):

$$r_n a_n x^{d_n} = 0 x^{d_n}$$

Since $a_n \neq 0$, $r_n = 0$.

(1) becomes $r_1 p_1 + \dots + r_{n-1} p_{n-1} = \bar{0}$

Now repeat to get $r_{n-1} = 0$,

and hence $r_1 = \dots = r_n = 0$. ∎

$\mathbb{P}$ space of all polynomials has infinite dimension.

$\mathbb{F}[a,b]$ If $p(x) \in \mathbb{P}$, consider

$p_{[a,b]} = p$ restricted to $[a,b]$.

$p_{[a,b]} \in \mathbb{F}[a,b]$. Ie, $\mathbb{P} \subsetneqq \mathbb{F}[a,b]$

so $\mathbb{F}[a,b]$ is infinite-dimensional.

Ex. $U = \{ f(x) \in \mathbb{F}(-\infty,\infty) : f'(x) = 3f(x) \}$

U is a subspace of $\mathbb{F}$.

$\bar{0}(x) = 0 \quad \bar{0}'(x) = \bar{0} = 3\bar{0}$ so $\bar{0} \in U$.

Let $f, g \in U$. $(f+g)'(x) = f'(x) + g'(x)$

$= 3f(x) + 3g(x)$

$= 3(f+g)(x)$

So $f+g \in U$

$c \in \mathbb{R} \quad (cf)'(x) = c(f'(x))$

$= c\, 3f(x)$

$= 3(cf)(x)$

Thus $cf \in U$.

We know: if $f'(x) = 3f(x)$

$f(x) = Ce^{3x}$.

Thus $\{e^{3x}\}$ is a basis for U, and $\dim(U) = 1$.