Some logical Reasoning

Incorrect Reasoning:

7 is a prime number

Therefore all odd numbers are prime.

Correct Reasoning:

For all prime numbers p, either $p=2$ or p is odd.

Therefore, 7 is an odd number.

For all $x \in \mathbb{R}$,
$$r_1 \cos(x) + r_2 \sin(x) = \bar{0}$$

Put in particular values of x to see what the eqn tells us about r_1, r_2.

Lecture 6: more on linear independence

Remark: In a vector space V, if $\bar{0} \neq u \in \text{Span}\{v_1, v_2, \dots, v_n\}$ then $\{u, v_1, \dots v_n\}$ is not independent.

For: $u = a_1 v_1 + \dots + a_n v_n$ for some $a_1, \dots, a_n$ not all 0

$$a_1 v_1 + \dots + a_n v_n - u = \bar{0}$$

Since not all coeffs are zero, $\{u, v_1, \dots v_n\}$ are not independent.

Theorem (Fundamental)

Let V be a vector space and suppose that V can be spanned by n vectors. Then any independent set of vectors in V has at most n elements.

Proof. Let $V = \text{span}\{v_1, \ldots, v_n\}$.

Let $\{u_1, \ldots, u_m\}$ be an independent set in V. Need to ~~sh~~ show $m \le n$.

Steinitz Exchange Lemma:

m of the vectors $v_1, \ldots, v_n$ can be replaced by $u_1, \ldots, u_m$ and the resulting set still spans V.

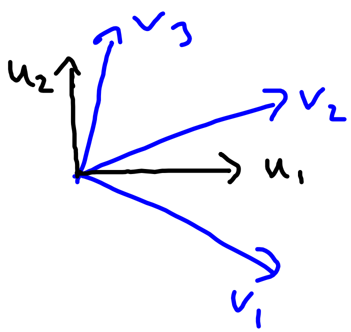
<u>pf.</u> Since $u_1 \in \text{span}\{v_1, \ldots, v_n\}$, then $u_1 = a_1 v_1 + \ldots + a_n v_n$ for some $a_1, \ldots, a_n$ not all 0. We may assume $a_1 \neq 0$. Claim $\{u_1, v_2, v_3 \ldots v_n\}$ spans V. For, if

$$w = r_1 v_1 + \ldots + r_n v_n \quad \text{then}$$

$$w = r_1 \frac{1}{a_1}(u_1 - a_2 v_2 + \ldots - a_n v_n) + r_2 v_2 + \ldots + r_n v_n$$

$$= \frac{r_1}{a_1} u_1 + \left(r_2 - \frac{r_1 a_2}{a_1}\right) v_2 + \ldots (\quad) v_n$$

$\mathbb{R}^2$



$\mathbb{R}^2 = \text{span}\{u_1, v_2, v_3\}$

$\text{span}\{u_1, v_2, v_3\} = \mathbb{R}^2$

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Now repeat.

$$u_2 = b_1 u_1 + b_2 v_2 + \dots + b_n v_n$$

not all b_i are 0.

Because $\{u_2, u_3\}$ is independent, at least one of $b_2, \dots, b_n$ is non-zero

WMA $b_2 \neq 0$; by the same argument,

$\text{span}\{u_1, u_2, v_3, \dots, v_n\} = V$.

Repeat until we have $\{u_1, u_2, \dots u_m, v_{m+1}, \dots v_n\}$ spans V.

Suppose $m > n$.

Then all vectors $v_1, \dots, v_n$ are replaced by $u_1, \dots, u_n$ and there is another u_{n+1} in the ind. set.

Then $u_{n+1} \in \text{span}\{u_1, \dots, u_n\} = V$

Hence $\{u_1, \dots, u_n, u_{n+1}\}$ is not independent.

This is a contradiction.

Hence $m \le n$. ∎

Defn. Suppose $V = \text{span}\{e_1, \dots, e_n\}$ and $\{e_1, \dots, e_n\}$ is independent. Then $\{e_1, \dots, e_n\}$ is a **basis** for V.

Theorem (Invariance)

Let $\{e_1, \dots, e_n\}$, $\{f_1, \dots, f_m\}$ be two different bases for V. Then $n = m$.

Pf. By Fund Thm, with $\{e_1, \dots, e_n\}$ as spanning set, $\{f_1, \dots, f_m\}$ as the ind set, $m \leq n$.

By Fund Thm with roles reversed $n \leq m$. Hence $n = m$. ∎

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