Lecture 5: Linear Combinations and Linear Independence

Continue with Ex

$$U = \{ (a_{ij}) \in M_{22} : a_{22} = 0 \}$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

$\in U$

Find span $\{A, B, D\}$.

For what a, b, c can we solve

$$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = r_1 A + r_2 B + r_3 D \quad ?$$

$$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} r_1 & r_1 + r_2 + 3r_3 \\ r_2 + 3r_3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

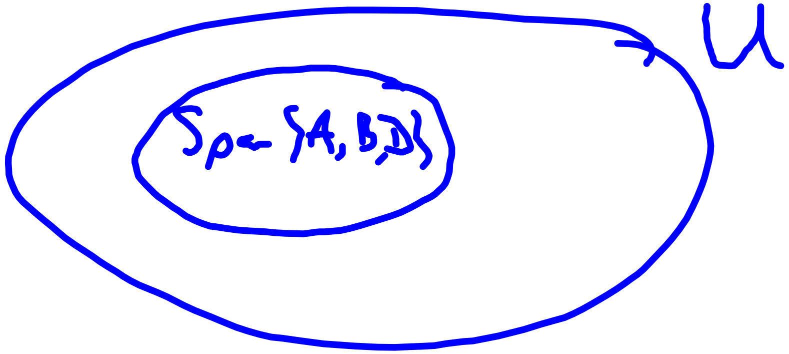
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 1 & 1 & 3 & b \\ 0 & 1 & 3 & c \end{array}\right) \xrightarrow[\text{elimination}]{\text{gaussian}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 1 & 1 & 3 & b \\ 0 & 0 & 0 & c+a-b \end{array}\right)$$

Soln exists if $c+a-b=0$

$$\text{Span}\{A,B,D\} = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \in U : c+a-b=0 \right\}$$

U 3 free choices

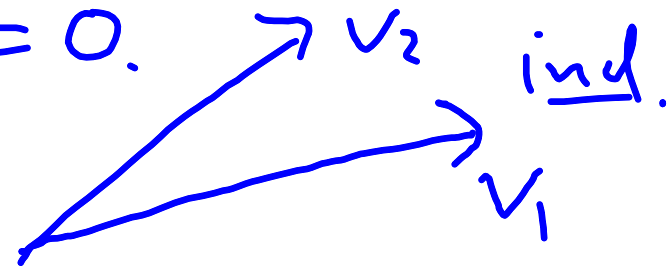
Span$\{A,B,D\}$ 2 free choices

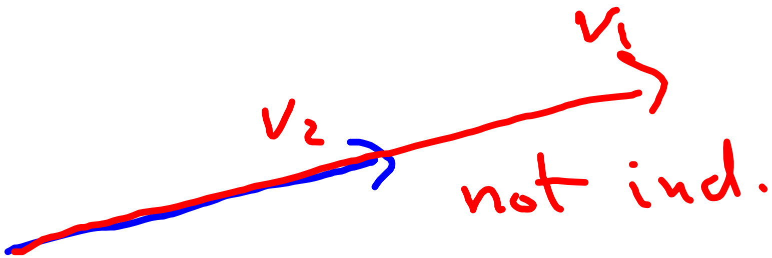


Defn. Let $v_1, \dots, v_n$ be vectors in a vector space. The set $\{v_1, \dots, v_n\}$ is (linearly) <u>independent</u> if the only solution to the equation

$$s_1 v_1 + s_2 v_2 + \dots + s_n v_n = \bar{0}$$

is $s_1 = s_2 = \dots = s_n = 0$.



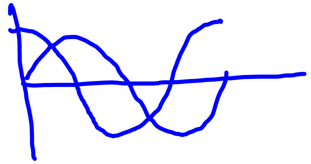


Ex. $\mathbb{F}[0, 2\pi]$

$\{\sin(x), \cos(x)\}$ is independent.

Suppose

$r_1 \sin(x) + r_2 \cos(x) = \bar{0}$



Consider $x = 0$.

$$r_1 \sin(0) + r_2 \cos(0) = 0$$
$$r_1 0 + r_2 1 = 0$$
$$\underline{r_2 = 0}.$$

$r_1 \sin(x) = \bar{0}$

Consider $x = \frac{\pi}{2}$.

$$r_1 \sin\left(\frac{\pi}{2}\right) = 0$$
$$\underline{r_1 = 0}.$$

Thus only solution is $r_1 = r_2 = 0$.

$\{\cos^2(x), \sin^2(x), f(x)\}$

where $f(x) = 2$ for all x.

Can we solve $r_1 \cos^2(x) + r_2 \sin^2(x) + r_3 2 = 0$ with r_1, r_2, r_3 not all 0?

$r_1 = r_2 = 1 \quad , \quad r_3 = -\frac{1}{2}$.

$$\cos^2(x) + \sin^2(x) - \frac{1}{2}\, 2 = \bar{0}.$$

Not independent = dependent.

Ex. Let A be a 2×2 matrix with nonzero eigenvalues $\lambda_1 \neq \lambda_2$. Let v_1, v_2 be associated eigenvectors.

$$A v_1 = \lambda_1 v_1 \quad , \quad A v_2 = \lambda_2 v_2$$

Then $\{v_1, v_2\}$ is lin. ind.

Why? Solve $\quad r_1 v_1 + r_2 v_2 = \bar{0} \quad (1)$

Multiply by A: $\quad A(r_1 v_1 + r_2 v_2) = A\bar{0}$

$$r_1 A v_1 + r_2 A v_2 = \bar{0}$$

$$r_1 \lambda_1 v_1 + r_2 \lambda_2 v_2 = \bar{0} \quad (2)$$

$(2) - \lambda_1 (1)$: $\quad 0 v_1 + r_2(\lambda_2 - \lambda_1) v_2 = \bar{0}$

$$\Rightarrow \underline{r_2 = 0}.$$

Substitute in (1): $\quad r_1 v_1 + 0 v_2 = \bar{0}$

$$\underline{r_1 = 0}.$$