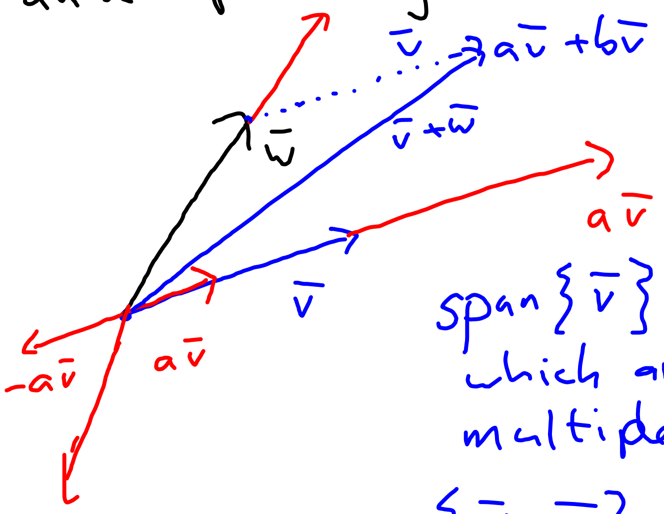
Lecture 4 Linear combinations and Spanning sets.

$\bar{v} \to a\bar{v} + b\bar{v} = (a+b)\bar{v}$



$\text{span}\{\bar{v}\}$ = all vectors which are scalar multiples of $\bar{v}$

$\bar{v}+\bar{w} \in \text{span}\{\bar{v}, \bar{w}\}$

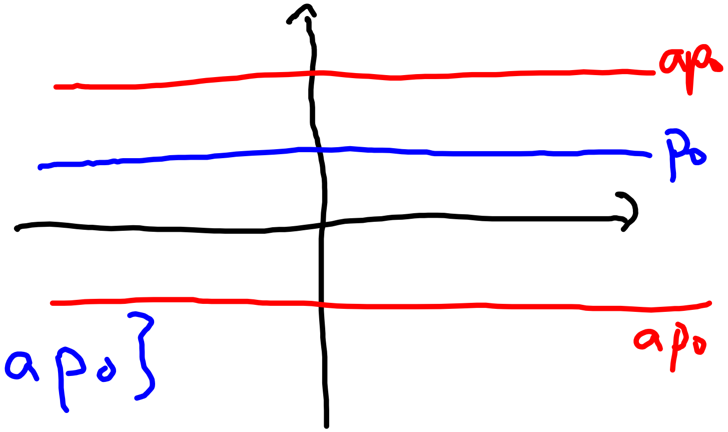
<u>Defn</u> Let $v_1, v_2, \dots, v_n$ be vectors in a vector space V. A <u>linear combination</u> of $v_1, \dots, v_n$ is a vector $u = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$, where $a_1, a_2, \dots, a_n \in \mathbb{R}$.

$\text{span}\{v_1, \dots, v_n\} = \{u \in V : u$ is a linear combination of $v_1, \dots, v_n\}$

Ex. $V = \mathbb{P}$ = space of polynomials

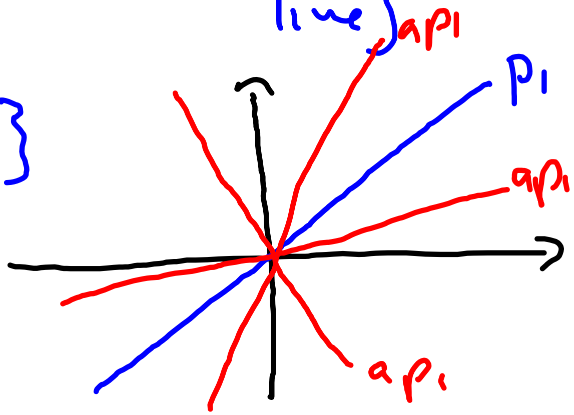
$p_0 = 1 = x^0$

$p_1 = x$



$\operatorname{span}\{p_0\} = \{p : p = a p_0\}$

$= \{p :$ the graph of p is a horizontal line$\}$



$\operatorname{span}\{p_1\} = \{p : p = ax\}$

$= \{p :$ graph of p is a line through $0\}$

(except vertical line)

$\operatorname{span}\{p_1, p_2\} = \{p : p = a_0 1 + a_1 x\}$

$=$ all lines $= \mathbb{P}_1$

Theorem. Let $U = \operatorname{span}\{v_1, \ldots, v_n\}$, where $v_1, \ldots, v_n$ are vectors in a vector space V. Then

1) U is a subspace of V, and $v_1, \ldots, v_n \in U$.

2) If W is another subspace of V and $v_1, \ldots, v_n \in W$ then $U \subseteq W$.

Proof. 1) For any $i = 1, \ldots, n$,

$$v_i = 1v_i = 0v_1 + 0v_2 + \ldots + 1v_i + \ldots 0v_n$$

So $v_i \in \operatorname{span}\{v_1, \ldots, v_n\} = U$.

$$\bar{0} = 0v_1 + \ldots + 0v_n \in U.$$

Let $v = a_1v_1 + \ldots + a_nv_n$

$w = b_1v_1 + \ldots + b_nv_n \in U$.

$$v + w = (a_1+b_1)v_1 + \ldots + (a_n+b_n)v_n \in U.$$

$$rv = (ra_1)v_1 + \ldots + ra_nv_n \in U.$$

By subspace test, U is a subspace.

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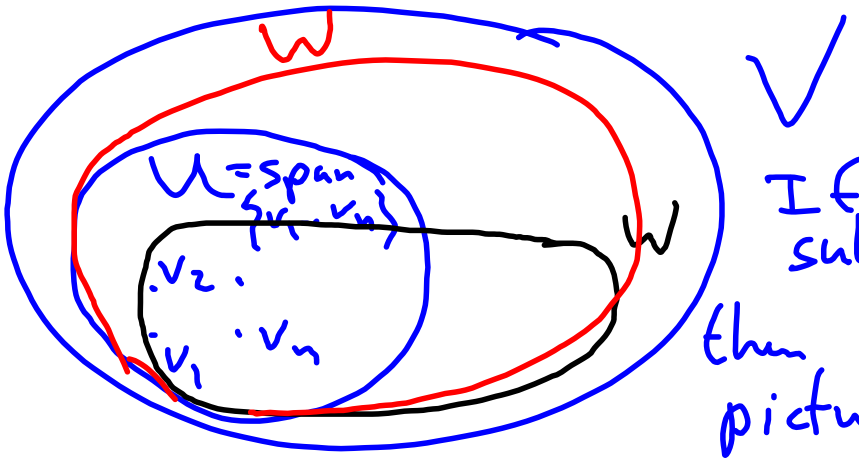
2) Let W be a subspace, with $v_1, \ldots, v_n \in W$.

Let $u \in U$. Then $u = a_1 v_1 + \ldots + a_n v_n$ for some $a_1, \ldots, a_n \in \mathbb{R}$.

Because W is a subspace, $u \in W$.

Thus $U \subseteq W$.



If W subspace then this picture not possible.

Ex. Let $p_1 = 1 + x$,
$p_2 = x + x^2$
$p_3 = 1 + 3x^2$

Find $\operatorname{span}\{p_1, p_2, p_3\} = U$.

$U \subseteq \mathbb{P}_2$, clearly!

Is $U = \mathbb{P}_2$? That is, for any $q = a + bx + cx^2 \in \mathbb{P}_2$, can q be written as $q = r_1 p_1 + r_2 p_2 + r_3 p_3$?

i.e. for all a, b, c, can I solve the eqn $a + bx + cx^2 = r_1(1+x) + r_2(x+x^2) + r_3(1+3x^2)$ for r_1, r_2, r_3?

Solve:
$$a = r_1 + r_3$$
$$b = r_1 + r_2$$
$$c = r_2 + 3r_3$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Yes, if and only if $(\quad)$ is invertible.

<u>Ex</u>. $U = \{ (a_{ij}) \in M_{22} : a_{22} = 0 \}$

U is a subspace.

Consider $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$C = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} \in U.$$

$$\text{span}\{A, B, C\} = \{ M \in U : M = r_1 A + r_2 B + r_3 C \}$$

For any $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$, do there exist r_1, r_2, r_3

st. $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = r_1 A + r_2 B + r_3 C$?

Compare coeffs in the two matrices

$$a = r_1 + r_3 \qquad c = r_2 + 3r_3$$

$$b = r_1 + r_2$$

Same as above example.