Lecture 35 : Principal Axis Thm & Review

Theorem. $T: V \to V$ liner operator on inner product space V. V finite dimension. V has an orthogonal basis of eigenvectors of T if and only if T is symmetric.

Proof "$\Longrightarrow$" Assume V has a basis $B = \{b_1 \ldots b_n\}$ of orthogonal eigenvectors. Then $T(b_i) = \lambda_i b_i$.

$$M_B(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$ is diagonal, so in particular is symmetric.

B can be converted to an orthonormal basis by ~~dividing~~ normaling each vector to have length 1. The matrix will be the same. By Thm (equivalents to defn of symmetric) $M_B(T)$ symmetric implies T is symmetric.

"$\Leftarrow$" Assume T is symmetric.

Proof is by induction on $n = \dim(V)$.

Base case: $\dim(V) = 1$.

Let $V = \text{span}\{b_1\}$. $\{b_1\}$ is automatically orthogonal, and $T(b_1) \in V$, so $T(b_1) = \lambda_1 b_1$.

Thus b_1 is an eigenvector.

Induction Hypothesis: For any vector space of dimension n, and any linear operator on it, if the operator is symmetric then the space has an orthogonal basis of eigenvectors.

Assume V has dimension $n+1$.

Since T is symmetric, all of its eigenvalues are real. Let λ_1 be one of the eigenvalues, b_1 an assoc. eigenvector.

Let $U = \text{span}\{b_1\}$. Then U is a T-invariant subspace.

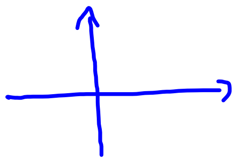
By yesterday's theorem, $U^\perp$ is also T-invariant. Thus $T: U^\perp \to U^\perp$ is a symmetric linear operator. $\dim(U^\perp) = n$, so by Induction Hypothesis, $U^\perp$ has an orthogonal basis of eigenvectors of T; $\{b_2, \ldots, b_{n+1}\}$. Since b_1 is orthogonal to all of $U^\perp$, $\{b_1, \ldots b_{n+1}\}$ is an orthogonal set of eigenvectors.
By orthogonality $\{b_1, \ldots, b_{n+1}\}$ is independent, and hence is the required basis for V.

Hence by induction, the statement is true for all spaces of all finite dimensions. $\blacksquare$

Chapter 6: Vector Spaces.

Key Idea: dimension.

$\mathbb{R}^2$



"is bigger than"

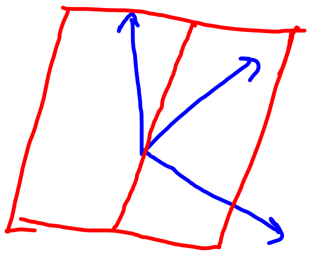
$\mathbb{R}^1$

$\dim(\mathbb{R}^2) = 2$ $\dim(\mathbb{R}^1) = 1$

$\dim(\mathbb{P}_n) = n+1$ so $\mathbb{P}_n$ is bigger than $\mathbb{R}^n$

basis as "minimal spanning set"
or as "maximal independent set"

Subspaces



In $\mathbb{P}_2$ subspace of odd polynomials
subspace of polys s.t. $p(3)=0$