Lecture 33 Symmetric Operators

$T: V \to V$ symmetric if

$$\langle v, T(w) \rangle = \langle T(v), w \rangle \quad \text{for all } v, w \in V.$$

<u>Ex</u>. $V = \mathbb{P}$ polynomials.

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)\, q(x)\, dx$$

$$T: \mathbb{P} \to \mathbb{P} \qquad T(p(x)) = x\,p(x)$$

T symmetric: $\langle p(x), T(q(x)) \rangle$

$$\cancel{\int_0^1}$$

$$= \langle p(x), x\,q(x) \rangle$$

$$= \int_0^1 p(x)\, x\, q(x)\, dx$$

$$= \int_0^1 x\,p(x)\, q(x)\, dx$$

$$= \langle T(p(x)), q(x) \rangle.$$

$S : \mathbb{P} \to \mathbb{P} \qquad S(p(x)) = p(x-1)$

S not symmetric: consider $p(x) = x$
$q(x) = x^2$

$\cancel{\langle p(x), S(q(x))\rangle}$

$$\langle p(x), S(q(x))\rangle = \langle x, S(x^2)\rangle$$
$$= \langle x, (x-1)^2\rangle$$
$$= \int_0^1 x(x-1)^2\,dx = \frac{1}{12}$$

$$\langle S(p(x)), q(x)\rangle = \langle x-1, x^2\rangle$$
$$= \int_0^1 (x-1)x^2\,dx = -\frac{1}{12}$$

<u>Pf</u> of theorem.

1) $\Rightarrow$ 2) Assume $\langle v, T(w)\rangle = \langle T(v), w\rangle$ for all $v, w \in V$.

Let $B = \{b_1, \dots b_n\}$ be an orthonormal basis for V.

$$M_B(T) = \left(c_B T(b_1) \; \dots \; c_B T(b_n) \right)$$

$$(M_B(T))_{ij} = \frac{\langle b_i, T(b_j)\rangle}{\|b_i\|^2} = \langle b_i, T(b_j)\rangle$$

T is symmetric
$$= \langle T(b_i), b_j\rangle$$

$\langle\,,\rangle$ commutative
$$= \langle b_j, T(b_i)\rangle$$
$$= (M_B(T))_{ji}$$

So $M_B(T)$ is symmetric.

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2) $\Rightarrow$ 3) immediate.

3) $\Rightarrow$ 4) Assume there is some orthonormal basis, $B = \{b_1 \dots b_n\}$ st. $M_B(T)$ is symmetric.

We need to show that $\langle b_i, T(b_j)\rangle = \langle T(b_i), b_j\rangle$ for all i,j.

By assumption $(M_B(T))_{ij} = (M_B(T))_{ji}$

$(M_B(T))_{ij} = \langle b_i, \cancel{T}(b_j)\rangle$

$(M_B(T))_{ji} = \langle b_j, T(b_i)\rangle = \langle T(b_i), b_j\rangle$

Equality on the left give equality on the right.

4) $\Rightarrow$ 1) Assume there is some orthonormal basis $B = \{b_1, \dots, b_n\}$ st. $\langle b_i, T(b_j)\rangle = \cancel{\langle b_j, T(b_i)\rangle}$

$= \langle T(b_i), b_j\rangle$

for all i,j.

Let $v, w \in V$; $v = r_1 b_1 + \dots + r_n b_n$

$w = s_1 b_1 + \dots + s_n b_n$

$$\langle v, T(w)\rangle = \langle r_1 b_1 + \dots + r_n b_n, T(s_1 b_1 + \dots + s_n b_n)\rangle$$

$$= \left\langle \sum_i r_i b_i, \sum_j s_j T(b_j) \right\rangle$$

$$= \sum_i \sum_j \langle r_i b_i, s_j T(b_j)\rangle$$

$$= \sum_i \sum_j r_i s_j \langle b_i, T(b_j)\rangle$$

$$= \sum_i \sum_j r_i s_j \langle T(b_i), b_j\rangle \quad \text{by assumption}$$

$$= \sum_i \sum_j \langle r_i T(b_i), s_j b_j\rangle$$

$$= \left\langle \sum_i r_i T(b_i), \sum_j s_j b_j \right\rangle$$

$$= \left\langle T\left(\sum_i r_i b_i\right), \sum_j s_j b_j \right\rangle$$

$$= \langle T(v), w\rangle$$