Lecture 31 Best choice for orthogonal basis

Theorem Let $T: V \to V$ be a linear operator. Then V has a basis of eigenvectors of T if and only if there is a basis B of V such that $M_B(T)$ is diagonal. $B = \{b_1, \dots, b_n\}$

Proof. $M_B(T) = \left(c_B T(b_1) \ \dots \ c_B T(b_n) \right)$

First suppose $\{b_1, \dots, b_n\}$ are eigenvectors of T.

$$T(b_i) = \lambda_i b_i = 0b_1 + 0b_2 + \dots + \lambda_i b_i + 0b_{i+1} \dots$$

$$c_B(T(b_i)) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \end{pmatrix} \leftarrow i\text{th row}$$

Then $M_B(T) = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \ddots & \lambda_n \end{pmatrix}$

Suppose $M_B(T)$ is diagonal; $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

then $c_B(T(b_i)) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_i \\ 0 \\ \vdots \end{pmatrix}$

So $T(b_i) = 0b_1 + \dots + 0b_{i-1} + \lambda_i b_i + \dots 0b_n$

$T(b_i) = \lambda_i b_i$

That is, b_i is an eigenvector. $\boxtimes$

Defn. $T : V \to V$ is diagonalisable if V has a basis of eigenvectors of T.

Observe: $B = \{b_1, \dots, b_n\}$ an orthogonal basis, $T : V \to V$

$$M_B(T) = \left(c_B T(b_1) \;\; \cdots \;\; c_B T(b_n) \right).$$

$$T(b_1) = \frac{\langle b_1, T(b_1) \rangle}{\|b_1\|^2} b_1 + \cdots + \frac{\langle b_n, T(b_1) \rangle}{\|b_n\|^2} b_n$$

$$c_B(T(b_1)) = \begin{pmatrix} \frac{\langle b_1, T(b_1) \rangle}{\|b_1\|^2} \\ \vdots \\ \frac{\langle b_n, T(b_1) \rangle}{\|b_n\|^2} \end{pmatrix}$$

If B is both orthogonal and consists of eigenvectors then

$$\langle b_i, T(b_j) \rangle = 0 \quad \text{for } i \neq j$$

and $$\frac{\langle b_i, T(b_i) \rangle}{\|b_i\|^2} = \lambda_i.$$

When does V have an orthogonal basis of eigenvectors of T?

Principal Axis Theorem Let $T: V \to V$ be a linear operator on the finite dimensional inner product space V. Then V has an orthogonal basis of eigenvectors of T if and only if T is symmetric.

Symmetric: matrix A is symmetric iff $A = A^T$

$\Leftrightarrow$ for $i = 1, \dots, n$ ith column of A = ith column of A^T

$\Leftrightarrow \forall i$, ith column of A = ith row of A

$\Leftrightarrow$ for all $i = 1, \dots, n$, for all $j = 1, \dots, n$ $\Leftrightarrow$ jth entry of ith col. of A = jth entry of ith row of A

$\Leftrightarrow$ $A = (a_{ij})$ $a_{ji} = a_{ij}$ for all i, j.

Let $\{e_1 \dots e_n\}$ standard basis for $\mathbb{R}^n$

$$e_i \cdot Ae_j = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot (a_{ij}) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow \text{jth row}$$

$$= \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

$$e_1 \cdot Ae_j = a_{1j}$$

$$e_i \cdot Ae_j = a_{ij}$$

And: $Ae_i \cdot e_j = a_{ji}$.

Thus A is symmetric if and only if

$$Ae_i \cdot e_j = e_i \cdot Ae_j \quad \text{for all } i, j.$$

<u>iff</u> $Av \cdot w = v \cdot Aw$ for any $v, w \in \mathbb{R}^n$

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Defn $T: V \to V$ be a linear operator on an inner product space. Then T is symmetric if for all $v, w \in V$

$$\langle v, Tw \rangle = \langle Tv, w \rangle.$$

Theorem V fin dim inner product space, $T: V \to V$ lin operator. Then the following are equivalent:

1) $\langle v, Tw \rangle = \langle Tv, w \rangle$ for all $v, w \in V$
2) The matrix of T w.r.t. every orthonormal basis of V is symmetric
3) The matrix of T w.r.t. some orthonormal basis is symmetric.
4) There is some orthonormal basis $B = \{b_1, \ldots, b_n\}$ of V st.

$$\langle b_i, T(b_j) \rangle = \langle T(b_i), b_j \rangle$$

for all i, j.