In $\mathbb{R}^3$ the vector equation of a plane is

$$n\cdot(p-p_0)=0$$

so a plane thru the origin

$$\bar{n}\cdot p=0 \qquad p=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

so a plane is really just

$$P = \{p\in\mathbb{R}^3 : p\cdot\bar{n}=0\}$$
$$= \{p\in\mathbb{R}^3 : \text{orthogonal to } n\}$$

<u>Def.</u> Let V an inner product space and U a subspace of V the <u>orthogonal complement</u> of U in V

$$U^\perp = \{v\in V : \langle v,u\rangle = 0 \text{ for all } u\in U\}$$

<u>Examples</u>

- $\mathbb{R}^3$ $\quad U=\operatorname{span}\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$

$$U^\perp = \left\{\begin{pmatrix}a\\b\\c\end{pmatrix}\in\mathbb{R}^3 : \begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}=0\right\}$$
$$= \left\{\begin{pmatrix}a\\b\\c\end{pmatrix} : a+2b+c=0\right\}$$
$$= \left\{\begin{pmatrix}a\\b\\c\end{pmatrix} : c=-a-2b\right\}$$
$$= \left\{\begin{pmatrix}a\\b\\-a-2b\end{pmatrix}\right\} = \left\{a\begin{pmatrix}1\\0\\-1\end{pmatrix}+b\begin{pmatrix}0\\1\\-2\end{pmatrix}\right\}$$
$$= \operatorname{span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\-2\end{pmatrix}\right\}$$

$\cdot\ \mathbb{P}_2 \qquad \langle p, q\rangle = p(0)q(0) + p(1)q(1) + p(-1)q(-1)$

$U = \operatorname{span}\{x^2+1,\ x-1\}$

$U^\perp = \{\underbrace{ax^2+bx+c}_{p} \in \mathbb{P}_2 : \langle p, x^2+1\rangle = \langle p, x-1\rangle = 0$

$= \{ax^2+bx+c : \begin{array}{l} c\cdot 1 + (a+b+c)\cdot 2 + (a-b+c)\cdot 2 = 0 \\ c\cdot -1 + (a+b+c)\cdot 0 + (a-b+c)\cdot -2 = 0 \end{array}$

$= \{ax^2+bx+c : \begin{array}{l} 4a+5c=0 \\ -2a+2b-3c=0 \end{array}\}$

$c = -4/5\, a$

$-2a+2b-3(-4/5\, a) = 0$

$2/5\, a + 2b = 0$

$b = -1/5\, a$

$= \{ax^2 - \frac{1}{5}ax - 4/5\, a\}$

$= \operatorname{span}\{x^2 - 1/5\, x - 4/5\}$

M_{22} $\quad \langle A, B \rangle = tr(AB^T)$

$$U = span\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

$$U^\perp = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\rangle = 0 \right\}$$

$$\left\langle \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle = tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right)$$

$$= tr\begin{pmatrix} a+b & b \\ c+d & d \end{pmatrix} = a+b+d$$

$$\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle = tr\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right)$$

$$= tr\begin{pmatrix} a & a \\ c & c \end{pmatrix} = a+c$$

$$a+b+d = 0 \longrightarrow \cancel{b=0} \;\; d = -a-b$$

$$a+c = 0 \longrightarrow \cancel{a} \;\; c = -a$$

$$U^\perp = \left\{ \begin{bmatrix} a & b \\ -a & -a-b \end{bmatrix} \right\} = span\left\{ \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$$

<u>Thm.</u> If V is an inner product space, U a subspace of V then

1) $U^\perp$ is a subspace of V

2) $U \oplus U^\perp = V$

<u>Proof</u>

1) • $0 \in U^\perp$, $\langle u, 0 \rangle = 0$ for all $u \in U$

• $u_1, u_2 \in U^\perp$, $\langle u_1 + u_2, u \rangle = \langle u_1, u \rangle + \langle u_2, u \rangle$

$= 0 + 0 = 0$

• so $u_1 + u_2 \in U^\perp$

• $v \in U^\perp$, $a \in \mathbb{R}$, $\langle av, u \rangle = a \langle v, u \rangle$

$u \in U$ $\qquad = a \cdot 0 = 0$

2) • $v \in U \cap U^\perp$, $v \in U^\perp \to \langle v, u \rangle = 0$ for all $u \in U$

but $v \in U$ so $\langle v, v \rangle = 0$

so $v = 0$

• Let $\{e_1, \dots, e_m\}$ be an orthogonal basis for U. Then if $v \in V$

$$v = v - \left(\frac{\langle v, e_1 \rangle}{\|e_1\|^2} e_1 + \dots + \frac{\langle v, e_m \rangle}{\|e_m\|^2} e_m \right)$$

orthogonal to $\{e_1, \dots, e_m\}$ (Gram-Schmidt)

$$+ \underbrace{\left(\frac{\langle v, e_1 \rangle}{\|e_1\|^2} e_1 + \dots + \frac{\langle v, e_m \rangle}{\|e_m\|^2} e_m \right)}_{\in U}$$

so $v - (\quad) \in U^\perp$

Title: Nov 16 - 12:26 PM (4 of 8)

Find the vector closest to v in the subspace U of an inner product space V.

In $\mathbb{R}^3$

$$\text{proj}_v u = \frac{u \cdot v}{\|v\|^2} v$$

In an inner product space V with subspace U

$$\text{proj}_U v = \frac{\langle v, e_1 \rangle}{\|e_1\|^2} e_1 + \frac{\langle v, e_2 \rangle}{\|e_2\|^2} e_2 + \ldots + \frac{\langle v, e_n \rangle}{\|e_n\|^2} e_n$$

where $\{e_1, \ldots, e_n\}$ is an <u>orthogonal</u> basis for U.

Approximation Thm. In an inner product V with subspace U. if $v \in V$ then

$$\mathrm{proj}_U v$$

is the closest vector to v in U in the sense that

$$\| v - \mathrm{proj}_U(v) \| \leq \| v - u \| \text{ for all } u \in U$$

Example $V = C[-1,1]$ $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$

Find the quadratic poly. closest to $f(x) = |x|$.

$$U = \mathbb{P}_2 = \mathrm{span}\{1, x, x^2\} \xrightarrow{G\text{-}S} \mathrm{span}\{1, x, 3x^2 - 1\}$$

$$\mathrm{proj}_U |x| = \frac{\langle |x|, 1 \rangle}{\|1\|^2} \cdot 1 + \frac{\langle |x|, x \rangle}{\|x\|^2} \cdot x + \frac{\langle |x|, 3x^2-1 \rangle}{\|3x^2-1\|^2} \cdot 3x^2 - 1$$

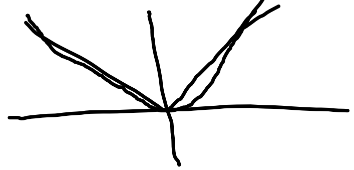
$$\langle |x|, 1 \rangle = \int_{-1}^{1} |x|\, dx = 2 \int_0^1 x\, dx = 1$$

$$\|1\|^2 = \int_{-1}^{1} 1 = 2$$

$$\langle |x|, x \rangle = \int_{-1}^{1} |x| \cdot x\, dx = 2 \int_0^1 x^2\, dx$$

$$\mathrm{proj}_U |x| = \frac{1}{2} \cdot 1 + 0 \cdot x + \frac{\frac{1}{2}}{8/5} \cdot 3x^2 - 1$$

$$= \frac{5}{16}(3x^2 - 1)$$



$C[a,b]$

$U = \{1, \sin x, \cos x, \sin 2x, \cos 2x, \dots\}$

The Fourier approx. of f is just $\mathrm{proj}_U f$