Proof:

$$e_{m+1} = v - t_1 e_1 - t_2 e_2 - \cdots - t_m e_m$$

where $t_k = \dfrac{\langle v, e_k \rangle}{\|e_k\|^2}$ for $k = 1, \ldots, m$

We need to check e_{m+1} is orth. to each $e_1, \ldots, e_m$.

Check if $\langle e_m, e_k \rangle = 0$, for $k = 1, \ldots, m$.

$$\begin{aligned}
\langle e_m, e_k \rangle &= \langle v - t_1 e_1 - \cdots - t_m e_m, e_k \rangle \\
&= \langle v, e_k \rangle - t_1 \langle e_1, e_k \rangle - \cdots - t_k \langle e_k, e_k \rangle - \cdots - t_m \langle e_m, e_k \rangle \\
&= \langle v, e_k \rangle - t_k \langle e_k, e_k \rangle \\
&= \langle v, e_k \rangle - \frac{\langle v, e_k \rangle}{\|e_k\|^2} \|e_k\|^2 = 0
\end{aligned}$$

So $\{e_1, \ldots, e_{m+1}\}$ is orthogonal.

Gram - Schmidt Orthogonalization:

Let V be an inner product space & let $\{v_1, \ldots, v_n\}$ be any basis of V. Define $e_1, \ldots, e_n$ as

$$e_1 = v_1$$

$$e_2 = v_2 - \frac{\langle v_2, e_1 \rangle}{\|e_1\|^2} \cdot e_1$$

$$e_3 = v_3 - \frac{\langle v_3, e_1 \rangle}{\|e_1\|^2} \cdot e_1 - \frac{\langle v_3, e_2 \rangle}{\|e_2\|^2} \cdot e_2$$

$$\vdots$$

$$e_k = v_k - \frac{\langle v_k, e_1 \rangle}{\|e_1\|^2} \cdot e_1 - \cdots \frac{\langle v_k, e_{k-1} \rangle}{\|e_{k-1}\|^2} \cdot e_{k-1}$$

for each $k = 1, \ldots, n$

Then
1) $\{e_1, \ldots, e_n\}$ is an orthogonal basis of V
2) $\operatorname{span}\{e_1, \ldots, e_k\} = \operatorname{span}\{v_1, \ldots, v_k\}$

<u>Proof</u>: (2) follows easily since each e_k is a linear combination of $\{v_1, \ldots, v_k\}$, and vice-versa.

So $\text{span}\{e_1, \ldots, e_n\} = \text{span}\{v_1, \ldots, v_n\}$ and since $\{v_1, \ldots, v_n\}$ is a basis of V, so is $\{e_1, \ldots, e_n\}$

$\{e_1, \ldots, e_n\}$ is orthogonal by the orthogonal lemma.

Ex 4: $V = \mathbb{P}_3$, with inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$.

Apply G-S orth. to the basis

$\{1, x, x^2, x^3\}$ of V.

Sol: $\boxed{e_1 = 1}$

$$e_2 = x - \frac{\langle x, 1 \rangle}{\|1\|^2} \cdot 1 = x - \frac{\int_{-1}^{1} x \cdot dx}{2} \cdot 1 \qquad \left\{ \begin{array}{l} 2 = \|1\|^2 \\ 2 = \int_{-1}^{1} 1 \cdot dx \end{array} \right\}$$

$$= x - \frac{\frac{1}{2}x^2 \Big|_{-1}^{1}}{2} \cdot 1$$

$$e_2 = x - \frac{\langle x, 1 \rangle \cdot 1}{\|e_1\|^2} = x - 0 = x$$

$\boxed{e_2 = x}$

$$e_3 = x^2 - \frac{\langle x^2, x\rangle}{\|x\|^2} \cdot x - \frac{\langle x^2, 1\rangle}{\|1\|^2} \cdot 1$$

$$e_3 = x^2 - \frac{\int_{-1}^{1} x^3 \cdot dx}{\int_{-1}^{1} x^2 \cdot dx} \cdot x - \frac{\int_{-1}^{1} x^2 \cdot dx}{\int_{-1}^{1} 1 \cdot dx} \cdot 1$$

$$e_3 = x^2 - \frac{0}{\frac{2}{3}} \cdot x - \frac{\frac{2}{3}}{2} \cdot 1$$

$$\boxed{e_3 = x^2 - \frac{1}{3}}$$

$$e_4 = x^3 - \frac{\langle x^3, 1\rangle}{\|1\|^2}\cdot 1 - \frac{\langle x^3, x\rangle}{\|x\|^2}\cdot x - \frac{\langle x^3, x^2 - \frac{1}{3}\rangle}{\|x^2 - \frac{1}{3}\|^2}\cdot\left(x^2 - \frac{1}{3}\right)$$

$$\frac{\int_{-1}^{1} x^3 \, dx}{\int_{-1}^{1} 1 \, dx} - \frac{\int_{-1}^{1} x^4 \, dx}{\int_{-1}^{1} x^2 \, dx} - \frac{\int_{-1}^{1} x^3 (x^2 - 1/3) \, dx}{\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 \, dx}$$

$$e_4 = \frac{1}{5}\left(5x^3 - 3x\right)$$