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Lecture 3 Subspaces

Defn ~~A subspace U of a vector space V~~
Let V be a vector space, and U be a subset of V, $U \subseteq V$. U is a subspace of V if it is itself a vector space, under the same operations as V.

Examples $\{0\}$, V are subspaces of the vector space V.
If $U \neq \{0\}$, $U \neq V$, U subspace of V we say it is a non-trivial, proper subspace.
$\mathbb{C}[a,b]$ is a subspace of $\mathbb{F}[a,b]$.
$\mathbb{D}[a,b]$ is a subspace of $\mathbb{C}[a,b]$
$\mathbb{P}$ is a subspace of $\mathbb{D}(-\infty,\infty)$.

Non-examples
$\mathbb{P}_2$ is not a subspace of $\mathbb{R}^3$ - not a subset
$M_{2,2}$ is not a subspace of $M_{2,3}$ - not a subset
the set of polynomials of degree exactly 3 is not a subspace of $\mathbb{P}_3$ - not itself a vector space, as does not have $\bar{0}$.

Subspace Test Let U be a subset of the vector space V. U is a subspace if and only if the following hold:

1) $\bar{0} \in \mathcal{U}$

2) $u_1, u_2 \in \mathcal{U} \Rightarrow u_1 + u_2 \in \mathcal{U}$

3) $u \in \mathcal{U} \Rightarrow au \in \mathcal{U}$ for all $a \in \mathbb{R}$.

~~Proof~~ ~~Why? Most of the axioms A1–8, S1–5 hold because the operations of U and V are the same. the rest 1, 2 say precisely A1, S1.~~

Proof A1 is 2).

A2, A3 hold because + in $\mathcal{U}$ and V are the same.

A4 is 1), and the fact that operations are the same.

A5 is by 3) and because $-v = (-1)v$.

S1 is 3)

S2 – S5 because operations in $\mathcal{U}$, V are the same. ∎

Example ~~If V is a vector space~~

$$\mathcal{U} = \{ \underset{(a_{ij})}{\overset{}{A}} \in M_{2,2} : a_{22} = 0 \}$$

Claim
$\mathcal{U}$ is a subspace of $M_{2,2}$:

Pf $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{U}$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathcal{U}$ then

$d = 0$ and $s = 0$.

thus $A+P = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}$ and $d+s=0$.

Hence $A+P \in U$.

If $A \in U$, $t \in \mathbb{R}$, $tA = \begin{pmatrix} ta & tb \\ tc & td \end{pmatrix}$ and $d=0$ $\Rightarrow td=0$.

thus $tA \in U$.

Hence U satisfies the conditions of the subspace test, so is a subspace.

<u>Example</u> $W = \{ \underset{(a_{ij})}{\overset{}{A}} \in M_{2,2} : a_{22} = 1 \}$.

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \notin W$, so W is not a subspace.

<u>Example</u> $W = \{ A = (a_{ij}) \in M_{2,2} : a_{11} a_{22} = 0 \}$.

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$.

Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in W$.

$A+P = \begin{pmatrix} a+p & b+q \\ c+r & d+s \end{pmatrix}$. $(a+p)(d+s) = ad + pd + as + ps$
$= pd + as = 0$?

Not necessarily. Say $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$

$A+P = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \notin W$.

<u>Example</u> $U = \{ f \in \mathcal{C}(-\infty,\infty) : f(1) = 0 \}$

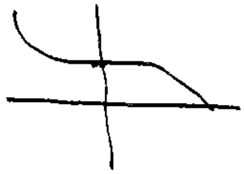
$\bar{0}(1) = 0$, so $\bar{0} \in U$.

Suppose $f, g \in U$. $(f+g)(1) = f(1) + g(1) = 0 + 0 = 0$
so $f+g \in U$.

$(af)(1) = a f(1) = a\,0 = 0$ so $af \in U$.

Thus U is a subspace of $\mathcal{C}(-\infty,\infty)$.

$W = \{ f \in \mathcal{C}(-\infty,\infty) : f(1) = 5 \}$



$\bar{0}(1) = 0 \neq 5$.

Also $(f+g)(1) = f(1) + g(1) = 5 + 5 = 10 \neq 5$.