$a_0 = 0,\ a_1 = 1,\ a_2 = -1$ in $\mathbb{P}_2$ $\qquad \dfrac{\prod_{i \neq k}(x - a_i)}{\prod_{i \neq k}(a_k - a_i)}$

$$\delta_0 = \frac{(x-1)(x+1)}{(0-1)(0+1)} = 1 - x^2$$

$$\delta_1 = \frac{(x-0)(x+1)}{(1-0)(1+1)} = \frac{x^2 + x}{2} = \frac{1}{2}x^2 + \frac{1}{2}x$$

$$\delta_2 = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{x^2 - x}{2} = \frac{1}{2}x^2 - \frac{1}{2}x$$

So $\{1 - x^2, \frac{1}{2}x^2 + \frac{1}{2}x, \frac{1}{2}x^2 - \frac{1}{2}x\}$ is orthnorm. basis for $\mathbb{P}_2$.

$p(x) = x^2 + 1$

$$p(x) = p(a_0)\,\delta_0(x) + p(a_1)\,\delta_1(x) + p(a_2)\,\delta_2(x)$$
$$= 1 \cdot (1 - x^2) + 2 \cdot \left(\tfrac{1}{2}x^2 + \tfrac{1}{2}x\right) + 2 \cdot \left(\tfrac{1}{2}x^2 - \tfrac{1}{2}x\right)$$

$$\langle \delta_k, \delta_k \rangle = \delta_k(a_0)^2 + \delta_k(a_1)^2 + \dots + \delta_k(a_n)^2$$

$$= 0 + 0 + \dots + \underbrace{1}_{k\text{-th}} + 0 + \dots + 0$$

$$= 1$$

Expansion Thm. gives us if $p(x) \in \mathbb{P}_n$

$$p(x) = \langle p, \delta_0 \rangle \delta_0(x) + \dots + \langle p, \delta_n \rangle \delta_n(x)$$

$$= p(a_0)\,\delta_0(x) + \dots + p(a_n)\,\delta_n(x)$$

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$\mathbb{P}_n$ - want to define inner product

Let $a_0, a_1, \ldots, a_n$ be real numbers.

<u>Define</u> $\langle p, q\rangle = p(a_0)q(a_0) + p(a_1)q(a_1) + \ldots + p(a_n)q(a_n)$

<u>Lagrange Polynomials</u>

Given $a_0, \ldots, a_n$ define

$$\delta_k = \frac{\prod_{i\neq k}(x-a_i)}{\prod_{i\neq k}(a_k-a_i)} = \frac{(x-a_0)(x-a_1)\ldots(x-a_n)}{(a_k-a_0)(a_k-a_1)\ldots(a_k-a_n)}$$

<u>Claim</u>: $\{\delta_0, \ldots, \delta_n\}$ forms an orthonormal basis for $\mathbb{P}_n$

$$\delta_k(a_i) = 0 \quad i \neq k$$

$$\delta_k(a_k) = 1$$

$$\langle \delta_k, \delta_i\rangle = \delta_k(a_0)\delta_i(a_0) + \ldots + \delta_k(a_n)\delta_i(a_n)$$

$$= 0\cdot 0 + \ldots + \underbrace{1\cdot 0}_{k\text{th}} + \ldots + \underbrace{0\cdot 1}_{i\text{-th}} + \ldots + 0\cdot 0$$

$$= 0$$

So $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -8 \end{pmatrix}, \begin{pmatrix} -19 \\ 10 \\ -1 \end{pmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$

$\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{77}} \begin{pmatrix} 2 \\ 3 \\ -8 \end{pmatrix}, \frac{1}{\sqrt{462}} \begin{pmatrix} -19 \\ 10 \\ -1 \end{pmatrix} \right\}$ is an orthonorm. basis for $\mathbb{R}^3$

<u>Theorem</u> If $\{e_1, \ldots, e_n\}$ is an orthogonal basis for V and $v \in V$ is any vector in V

$$v = \frac{\langle v, e_1 \rangle}{\|e_1\|^2} \cdot e_1 + \frac{\langle v, e_2 \rangle}{\|e_2\|^2} e_2 + \cdots + \frac{\langle v, e_n \rangle}{\|e_n\|^2} \cdot e_n$$

The coefficients $\frac{\langle v, e_1 \rangle}{\|e_1\|^2}, \ldots, \frac{\langle v, e_n \rangle}{\|e_n\|^2}$ are called <u>Fourier coefficients</u>.

Thm. Every orthogonal set is lin. ind.

Proof: Let $\{e_1, \dots, e_n\}$ be an orthogonal set. We write

$$a_1 e_1 + a_2 e_2 + \dots + a_n e_n = 0$$

WTS $a_1 = a_2 = \dots = a_n = 0$.

Well

$$\langle e_1, a_1 e_1 + \dots + a_n e_n \rangle = \langle e_1, 0 \rangle$$

$$a_1 \langle e_1, e_1 \rangle + a_2 \langle e_1, e_2 \rangle + \dots + a_n \langle e_1, e_n \rangle = 0$$

$$a_1 \|e_1\|^2 + a_2 \cdot 0 + \dots + a_n \cdot 0 = 0$$

$$a_1 \|e_1\|^2 = 0$$

$$a_1 = 0 \quad \text{since } \|e_1\| \neq 0$$

Similiarly, $\langle e_2, a_1 e_1 + \dots + a_n e_n \rangle = \langle e_2, 0 \rangle$

$$a_2 = 0$$

Thm. If $\{e_1, \ldots, e_n\}$ is orthogonal then $\left\{\frac{1}{\|e_1\|}e_1, \ldots, \frac{1}{\|e_n\|}e_n\right\}$ is orthonormal.

This process is called **normalizing**.

$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$, $\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\3\\-8\end{pmatrix}, \begin{pmatrix}-19\\10\\-1\end{pmatrix}\right\}$

$\left\{\begin{pmatrix}4\\-11\end{pmatrix}, \begin{pmatrix}-7\\-3\end{pmatrix}\right\}$, $\{\sin x, \cos x\}$

$\{\sin 2x, \cos 2x\}$

These are all lin. ind.

Def. A set of vectors $\{e_1, \ldots, e_n\}$ is called orthonormal if

1) $\{e_1, \ldots, e_n\}$ is orthogonal
2) $\|e_i\| = 1$ for all i

Given $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ -8 \end{pmatrix}, \begin{pmatrix} -19 \\ 10 \\ -1 \end{pmatrix} \right\}$ is orthogonal can we find a related orthonormal set?

Hint: Multiplication by a nonzero scalar does not change orthogonality.

- Multiply by $\dfrac{1}{\|v\|}$
- So $\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{77}} \begin{pmatrix} 2 \\ 3 \\ -8 \end{pmatrix}, \frac{1}{\sqrt{462}} \begin{pmatrix} -19 \\ 10 \\ -1 \end{pmatrix} \right\}$ is orthonorm.

Examples

- $\mathbb{R}^3$ with dot product

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}1\\2\\1\end{pmatrix},\begin{pmatrix}2\\3\\-8\end{pmatrix},\begin{pmatrix}-19\\10\\-1\end{pmatrix}\right\}$$

- $\mathbb{R}^2$ $A=\begin{pmatrix}2&1\\1&1\end{pmatrix}$ $\langle X,Y\rangle = X^T A Y$

$$\left\{\begin{pmatrix}4\\-11\end{pmatrix},\begin{pmatrix}7\\-3\end{pmatrix}\right\}$$

- $C[0,2\pi]$ $\langle f,g\rangle = \int_0^{2\pi} f(x)g(x)\,dx$

$\{\sin 2x, \cos 2x\}$

$\{\sin x, \cos x\}$

Recall in $\mathbb{R}^2$

$$u \cdot v = \|u\| \cdot \|v\| \cdot \cos\theta$$

so

$$u \perp v \text{ iff } u \cdot v = 0$$

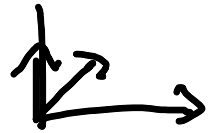
<u>Def.</u> Two nonzero vectors u and v in an inner product space V are called <u>orthogonal</u> if $\langle u, v\rangle = 0$.

A set of vectors $\{e_1, \ldots, e_n\}$ in V is an <u>orthogonal set of vectors</u> if

1) $e_i \neq 0$ for all i
2) $\langle e_i, e_j\rangle = 0$ for all $i \neq j$

Q. What are some of the properties of the standard basis in $\mathbb{R}^2$ that make it easy to work with?

- $\begin{bmatrix}0\\1\end{bmatrix} \perp \begin{bmatrix}1\\0\end{bmatrix}$ $\qquad$ $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$



- easy to write $\begin{bmatrix}a\\b\end{bmatrix}$ as a linear comb.
- both have length $\underline{1}$