Lecture 28: Fourier Series

$$\langle \, , \, \rangle : V \times V \longrightarrow \mathbb{R}$$

$$\| v \| = \sqrt{\langle v, v \rangle}$$

Schwarz Inequality:

$$\langle v, w \rangle \leq \| v \| \cdot \| w \|$$

$$(\; v \cdot w = \| v \| \, \| w \| \cos(\theta))$$

<u>Ex.</u> $\mathbb{F}[a,b]$.

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$$

Schwarz inequality:

$$\langle f, g \rangle^2 = \left(\int_a^b f(x) g(x) \, dx \right)^2$$

$$\| f \|^2 \, \| g \|^2 = \int_a^b (f(x))^2 dx \int_a^b (g(x))^2 dx$$

Defn We say v, w are **orthogonal** if ~~...~~ $\langle v, w\rangle = 0$.

If $\{v_1, \ldots, v_n\}$ is orthogonal, then it is linearly independent.

If $\{v_1, \ldots, v_n\}$ is an orthogonal basis for V, then for any $v \in V$,

$$v = \frac{\langle v, v_1\rangle}{\|v_1\|^2} v_1 + \cdots + \frac{\langle v, v_n\rangle}{\|v_n\|^2} v_n$$

$$\left(\|v_1\|^2 = \langle v_1, v_1\rangle \right)$$

Ex. $C[-\pi, \pi]$ space of continuous functions

Consider the orthogonal bases, and their span:

$$F_n = \operatorname{span}\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots, \sin(nx), \cos(nx)\}$$

$$\langle \sin(kx), \sin(mx)\rangle$$

$$= \int_{-\pi}^{\pi} \sin(kx)\sin(mx)\, dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2}\sin(kx-mx) + \frac{1}{2}\sin(kx+mx)\; dx$$

$$= \frac{1}{2}\left[-\frac{\cos(k-m)x}{k-m} - \frac{\cos(k+m)x}{k+m} \right]_{-\pi}^{\pi}$$

$$= 0.$$

Let $f(x) \in C[-\pi, \pi]$.

Define $a_0 = \langle f, 1\rangle = \int_{-\pi}^{\pi} f(x)\, dx$

$$a_k = \langle f, \cos(kx)\rangle = \int_{-\pi}^{\pi} f(x)\cos(kx)\, dx$$

$$b_k = \langle f, \sin(kx)\rangle = \int_{-\pi}^{\pi} f(x)\sin(kx)\, dx$$

$$f_n(x) = a_0 + \frac{a_1}{\|\cos(x)\|^2}\cos(x) + \frac{b_1}{\|\sin(x)\|^2}\sin(x)$$

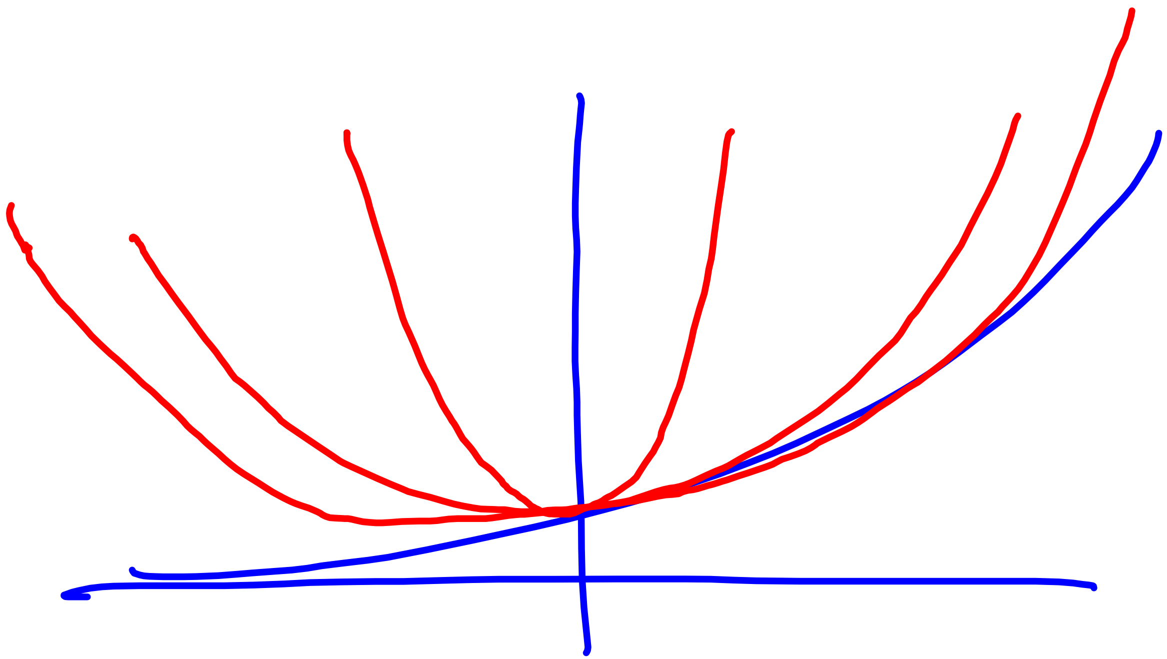
$$+ \cdots + \frac{a_n}{\|\cos(nx)\|^2}\cos(nx)$$

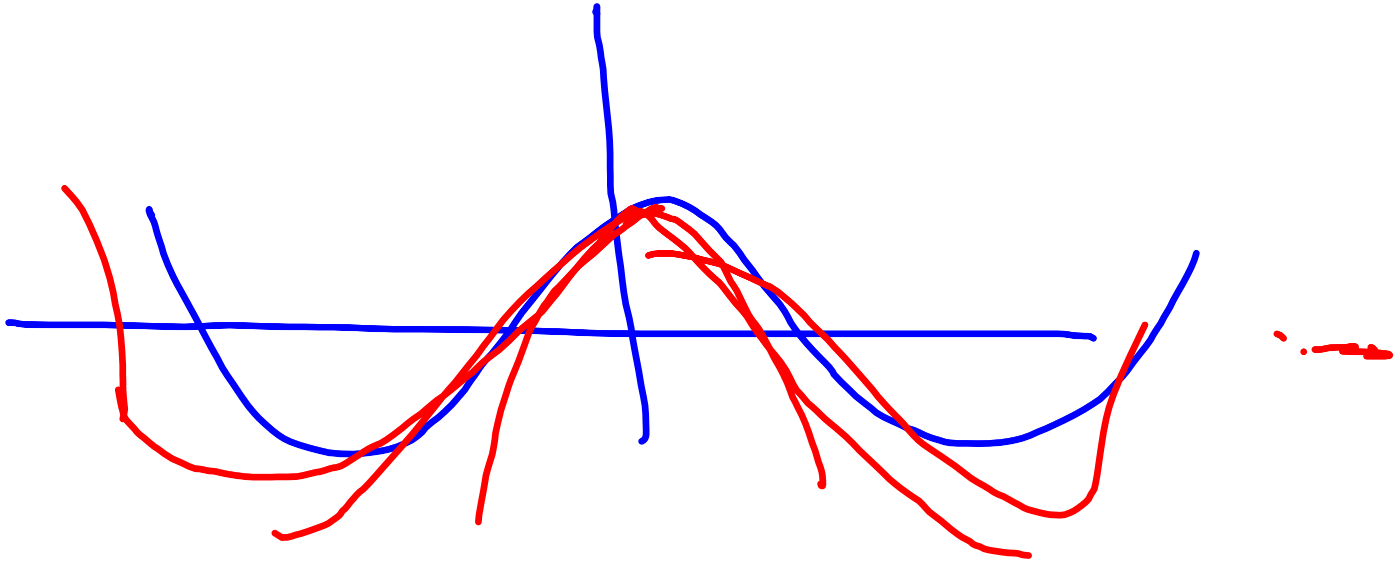
$$+ \frac{b_n}{\|\sin(nx)\|^2}\sin(nx).$$

$f_n(x) \in \mathbb{F}_n$.

$\| f(x) - f_n(x)\| \le \| f(x) - g(x)\|$ for any $g(x) \in \mathbb{F}_n$.

As $n \to \infty$, $f_n(x) \to f(x)$.





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