Lecture 27 Inner Product Spaces

Defn $\langle\, ,\, \rangle$ is an inner product on the vector space V if

(i) $\langle v, w\rangle = \langle w, v\rangle \quad \forall v, w \in V$

(ii) $\langle v+w, u\rangle = \langle v, u\rangle + \langle w, u\rangle$

(iii) $\langle rv, w\rangle = r\langle v, w\rangle$

(iv) $\langle v, v\rangle > 0$ for all $v \neq 0$ in V;
$\langle \vec{0}, \vec{0}\rangle = 0.$

$\langle\, ,\, \rangle : V \times V \longrightarrow \mathbb{R}$

$V \times V = \{(a,b) : a \in V,\ b \in V\}$

Analogously: $+ : V \times V \longrightarrow V$

scalar mult^n $* : \mathbb{R} \times V \longrightarrow V$

Ex 1) On $\mathbb{R}^n$

$$\langle v, w\rangle = v \cdot w = v^T w$$
$$= (v_1\ v_2 \ldots v_n) \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$
$$= v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$$

$\|v\|^2 = \langle v, v\rangle$

$u \cdot (v+w) = u \cdot v + u \cdot w$

$(ru) \cdot v = r(u \cdot v)$

$v \cdot v = v_1^2 + \ldots + v_n^2 > 0$

Thus the dot product is an inner product.

2) $V = \mathbb{R}^n$

Define $\langle v, w \rangle = v^T A w$, where A is an $n \times n$ matrix.

$\langle u+v, w \rangle$ ॥

$$(u+v)^T A w = (u^T + v^T) A w$$
$$= u^T A w + v^T A w$$
$$= \langle u, w \rangle + \langle v, w \rangle$$

Similarly $\langle av, w \rangle = a \langle v, w \rangle$

$$\langle v, v \rangle = v^T A v$$

<u>Theorem</u> $v^T A v > 0$ for all $v \in \mathbb{R}^n$ if and only if A is <u>positive definite</u>; all eigenvalues of A are real and positive.

If A is diagonal; $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$v^T A v = (v_1 \ldots v_n) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$
$$= (v_1 \ldots v_n) \begin{pmatrix} \lambda_1 v_1 \\ \lambda_2 v_2 \\ \vdots \\ \lambda_n v_n \end{pmatrix}$$
$$= \lambda_1 v_1^2 + \ldots + \lambda_n v_n^2$$

3) $C[a,b]$ - space of continuous functions on $[a,b]$

$$\langle f, g\rangle = \int_a^b f(x)\, g(x)\, dx$$

Verify this is an inner product:

1) symmetric: clear.

2) $\langle f+g, h\rangle = \int_a^b (f+g)h\, dx$

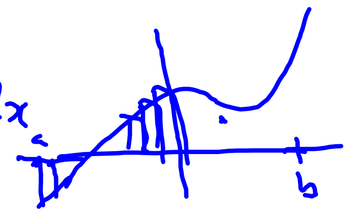
$= \int_a^b (fh+gh)\, dx$

$= \int_a^b fh\, dx + \int_a^b gh\, dx$

$= \langle f,h\rangle + \langle g,h\rangle.$

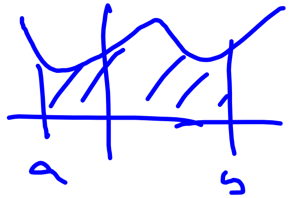
3) similar

4) $\langle f,f\rangle = \int_a^b (f(x))^2\, dx$



$\geq 0.$

because $f^2 \geq 0$ for all x.



Defn. Let $\langle\ ,\ \rangle$ be an inner product on V. (V is an inner product space.) We define the **norm** of a vector $v \in V$:

$$\|v\| = \sqrt{\langle v, v\rangle}$$

For $v, w \in V$, the **distance** from v to w is $d(v,w) = \|v-w\|$.

Schwarz Inequality

$$\langle v, w\rangle^2 \leq \|v\|^2 \|w\|^2$$

$T: V \to V$ / $T: \mathbb{R}^n \to \mathbb{R}^n$

$T(X) = AX$

Find a basis $B = \{b_1, \ldots, b_n\}$ so that

$b_1, \ldots, b_{r_1}$ a basis for G_{λ_1} $\quad Ab_i \in G_{\lambda_1}$

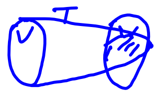
$b_{r_1}, \ldots, b_{r_2}$ - - - - G_{λ_2}

$\vdots$

Then ${}_B M_B(T) = [C_B T(b_1) \; \ldots \;]$

$$P^{-1} A P = \begin{pmatrix} \begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix} & 0 & 0 \\ \begin{matrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix}} & 0 \\ & 0 & \ddots \end{pmatrix}$$

$T: V \to W$



T is onto if $\text{im}(T) = W$

Explicitly: take $w \in W$. Ask: can I write w as $T(v)$ for some v?

Implicitly: find $\dim(\text{im}(T))$

or: find a basis for $\text{im}(T)$.

If $\dim(\text{im}(T)) \lneq \dim(W)$, then $\text{im}(T)$ is a proper subspace of W

$\dim(V) = \dim(\ker(T)) + \dim(\text{im}(T))$