$T: V \to W$ linear transformation.

$\ker(T) = \{ v \in V : T(v) = 0 \}$

$\operatorname{im}(T) = \{ w \in W : \text{there exists } a\ v \in V \text{ s.t. } T(v) = w \}$

<u>Ex</u>. $T: \mathbb{P}_1 \to \mathbb{R}^3$

$$T(p(x)) = \begin{pmatrix} p(0) \\ p(1) \\ p(-1) \end{pmatrix}$$

$$\ker(T) = \{ p(x) : T(p(x)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$$

$$= \{ p(x) = a + bx : \begin{pmatrix} a \\ a+b \\ a-b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$$

$$= \{ p(x) = 0 + 0x \}$$

$$\operatorname{im}(T) = \{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{R}^3 : \text{there exists } p(x) = a + bx \text{ st. } T(p(x)) = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \}$$

$$= \{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{R}^3 : \text{there exists } p(x) = a + bx \text{ st. } \begin{pmatrix} a \\ a+b \\ a-b \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \}$$

Find $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ so that there is a solution (a, b) to the equations

$$\begin{aligned} a &= u \\ a+b &= v \\ a-b &= w \end{aligned}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 0 & u \\ 1 & 1 & v \\ 1 & -1 & w \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & u \\ 0 & 1 & v-u \\ 0 & -1 & w-u \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 0 & u \\ 0 & 1 & v-u \\ 0 & 0 & w-u+v-u \end{array}\right)$$

$$\begin{aligned} 1a + 0b &= u \\ 0a + 1b &= v-u \\ 0a + 0b &= w-u+v-u \end{aligned}$$

A soln exists if $w-u+v-u = 0$. ie $w = 2u - v$.

$$\operatorname{im}(T) = \{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} : w = 2u - v \}.$$

$$= \{ \begin{pmatrix} u \\ v \\ 2u-v \end{pmatrix} : u, v, \cancel{w} \in \mathbb{R} \}$$

$$= \{ u \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + v \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} : u, v \in \mathbb{R} \}$$

$$= \operatorname{span} \{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \}$$

$\{F_1, \ldots F_k\}$ an orthogonal basis for a ~~vector space~~ subspace of $\mathbb{R}^n$

$(F_i \cdot F_j = 0$ for $i \neq j)$.

then

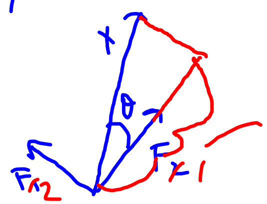
$$X = \frac{X \cdot F_1}{\|F_1\|} F_1 + \cdots + \frac{X \cdot F_k}{\|F_k\|} F_k$$

If $\{F_1, \ldots, F_k\}$ is orthonormal

$(F_i \cdot F_i = 1)$ then

$$X = (X \cdot F_1) F_1 + \ldots + (X \cdot F_k) F_k$$

Projections



$X \cdot F_1 = \|X\| \|F_1\| \cos\theta$

component of X in direction of F_1

$$X = \begin{pmatrix}\text{component of } X \\ \text{in direction of } F_1\end{pmatrix} F_1 + \begin{pmatrix}\text{component of } X \\ \text{in direction} \\ \text{of } F_2\end{pmatrix} F_2$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\text{comp. of } X}{\|X\|}$$

$$\frac{X \cdot F_1}{\|X\| \|F_1\|}$$

$$\text{comp. of } X = \frac{X \cdot F_1}{\|F_1\|}$$

$= $ projection of X onto F_1.

Definition An *inner product* on a vector space V is a function from $V \times V$ to $\mathbb{R}$ ~~V~~, denoted by $\langle \, , \, \rangle$ which satisfies:

(i) $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$

(ii) $\langle v+u, w \rangle = \langle v, w \rangle + \langle u, w \rangle$

(iii) $\langle av, w \rangle = a \langle v, w \rangle$, for any $a \in \mathbb{R}$

(iv) $\langle v, v \rangle \geq 0$, $\langle v, v \rangle = 0$ only if $v = \bar{0}$.