Announcements:

Midterm II Wed Nov 7 7- 8:30

REF 102

T13 120

Laurel office hours M: 2:30-4:00

HH 401

W: 3:30-4:30

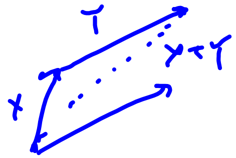
Math Helpcentre

Labs 9.4, 9.5 are deleted.

Lecture 25: Dot products and Orthogonality.

Review.

$$X, Y \in \mathbb{R}^n$$



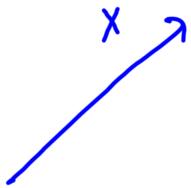
n=3. $X \times Y$ is a vector in 3-space.

Fundamentally a 3-dim concept.

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$X \cdot Y = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$X \cdot X = x_1^2 + x_2^2 + x_3^2$$



$$\| X \| = \sqrt{X \cdot X}$$

$$X, Y \in \mathbb{R}^n \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

$$X \cdot Y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

$$\boxed{X \cdot Y = X^T Y}$$

Dot product satisfies:

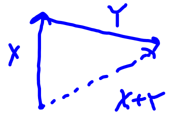
$$X\cdot Y = Y\cdot X \qquad \text{commutative}$$

$$X\cdot (Y+Z) = X\cdot Y + X\cdot Z \qquad \text{distributivity}$$

$$(aX)\cdot Y = a(X\cdot Y) = X\cdot (aY)$$

$$\|X\| = \sqrt{X\cdot X}$$

$$\|X+Y\| \le \|X\| + \|Y\| \qquad \text{triangle inequality}$$



$$|X\cdot Y| \le \|X\|\,\|Y\|$$

Cauchy Inequality

$$|X\cdot Y| = |x_1y_1 + x_2y_2 + \dots + x_ny_n|$$

$$\|X\|\,\|Y\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}\sqrt{y_1^2 + y_2^2 + \dots + y_n^2}$$

<u>Defn</u> Vectors $X, Y \in \mathbb{R}^n$ are <u>orthogonal</u> if $X\cdot Y = 0$.

$\Big($ $n=3$: $X\cdot Y = \|X\|\,\|Y\| \cos(\theta)$ $\quad X\cdot Y = 0 \iff \cos(\theta) = 0,\ \theta = \pi/2$ $\Big)$.

$\{X_1, \dots, X_k\}$ is <u>orthogonal</u> if $X_i \cdot X_j = 0$ for all $i \ne j$.

$\{X_1, \dots, X_k\}$ is <u>orthonormal</u> if orthogonal and $X_i \cdot X_i = 1$.

Pythagoras' Thm If $\{x_1, \dots, x_k\}$ is orthogonal then

$$\|x_1 + x_2 + \dots + x_k\|^2 = \|x_1\|^2 + \dots + \|x_k\|^2$$

Pf. $\|x_1 + \dots + x_k\|^2 = (x_1 + \dots + x_k)\cdot(x_1 + \dots + x_k)$

$$= x_1 \cdot (x_1 + \dots + x_k)$$
$$+ x_2 \cdot (x_1 + \dots + x_k)$$
$$+ \dots \quad x_k \cdot (x_1 + \dots + x_k)$$
$$= x_1 \cdot x_1 + 0 + \dots + 0 +$$
$$0 + x_2 \cdot x_2 + 0 + \dots 0 +$$
$$\dots \quad x_k \cdot x_k$$
$$= \|x_1\|^2 + \|x_2\|^2 + \dots + \|x_k\|^2$$

Theorem If $\{X_1, \dots, X_k\}$ is orthogonal then it is linearly independent.

Pf. Suppose $t_1 X_1 + \dots + t_k X_k = 0$.

Then $X_1 \cdot (t_1 X_1 + \dots + t_k X_k) = X_1 \cdot 0 = 0$

$$t_1 X_1 \cdot X_1 + t_2 \underbrace{X_1 \cdot X_2}_{0} + \dots + t_k \underbrace{X_1 \cdot X_k}_{0} = 0$$

$t_1 X_1 \cdot X_1 = 0$, by orthogonality

so $t_1 = 0$.

By repeating with i replacing 1 above, get $t_i = 0$ for each i.

Orthogonal Basis Let $\{F_1, \dots, F_k\}$ be an orthogonal basis for a subspace U of $\mathbb{R}^n$.

Let $X \in U$, then

$$X = \frac{X \cdot F_1}{\|F_1\|^2} F_1 + \dots + \frac{X \cdot F_k}{\|F_k\|^2} F_k .$$