Lecture 24: More examples of block diagonal form.

Finishing yesterday's example.

$T : \mathbb{P}_3 \to \mathbb{P}_3$

eigenvalues 1 with mult 3

-1 with mult 1

Basis $\left\{ \underbrace{\begin{pmatrix} 3 \\ -1 \\ 1 \\ 9 \end{pmatrix}}_{G_{-1}(T)}, \underbrace{\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}_{G_1(T)} \right\}$

$$M_E(T)\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$M_E(T)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$M_E(T)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \\ 2 \end{pmatrix} = 1\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Thus $M_B(T) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Example. $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} 3a+c \\ 2b \\ -a+3c \end{pmatrix}$$

$$M_E(T) = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Observe: $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector;

$U_1 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$ is a T-invariant subspace.

Claim: $U_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$ is T-invariant.

$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \in U_2$

$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \in U_2$.

$V = U_1 \oplus U_2$ ordered basis

$$B = \left\{\underset{b_1}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{b_2}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{b_3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}\right\}$$

~~$M_B(T) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$~~

$M_B(T) = \left(c_B T(b_1) \;\; c_B T(b_2) \;\; c_B T(b_3) \right)$

$T(b_1) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2b_1 + 0b_2 + 0b_3$

$$M_B(T) = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 3 & 1 \\ 0 & -1 & 3 \end{array}\right)$$

$T(b_2) = 3b_2 - 1b_3$

$T(b_3) = 1b_2 + 3b_3$

Instead $W_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ $\mathcal{D}$

$W_2 = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$M_{\mathcal{D}}(T) = \left(\begin{array}{cc|c} 3 & 1 & 0 \\ -1 & 3 & 0 \\ \hline 0 & 0 & 2 \end{array}\right)$$

Ex. $T : \mathbb{R}^3 \to \mathbb{R}^3$ $T(X) = AX$.

$$A = \begin{pmatrix} -3 & -1 & 0 \\ 4 & 1 & 3 \\ 4 & -2 & 4 \end{pmatrix}$$

$$c_A(y) = \det(yI - A)$$

$$= \det \begin{pmatrix} y+3 & 1 & 0 \\ -4 & y+1 & -3 \\ -4 & 2 & y-4 \end{pmatrix}$$

$$= (y-1)(y-1)(y+2)$$

eigenvalues 1 with mult 2

-2 with mult 1.

$G_{-2}(A) = \cancel{\ker(2I-2A)}\ \ker(-2I-A)$

$= \{ X : (-2I-A)X = 0 \}$

$$\begin{pmatrix} 1 & 1 & 0 \\ -4 & -1 & -3 \\ -4 & 2 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

solve the eqn: $G_{-2}(A) = \text{span}\left\{ \underset{b_1}{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}} \right\}$

$G_1(A) = \ker(I-A)^2.$

Solve $(I-A)X = 0$ (ie. find $\ker(I-A)$).

solns are span $\left\{ \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix} \right\}$.

Solve $(I-A)^2 X = 0$

solns are span $\left\{ \underset{b_2}{\begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}}, \underset{b_3}{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}} \right\}$

$Ab_1 = -2b_1 \quad Ab_2 = b_2 \quad Ab_3 = 1b_2 + 1b_3$

$$M_B(T) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Jordan Canonical Form

Let $T: V \to V$ be a linear operator.
Assume $\lambda_1 \ldots \lambda_k \in \mathbb{R}$ lists all the distinct eigenvalues of T. Then there is a basis B of V s.t. $M_B(T) = \mathrm{diag}(J_1 \ldots J_k)$ where each J_i is a Jordan block corresponding to λ_i.

$$J_i = \begin{pmatrix} \lambda_i & & 0 \\ & \ddots & \\ 0 & & \lambda_i \end{pmatrix}$$

with size the multiplicity of λ_i