Lecture 23: Block diagonal form

Midterm II Wed 7 Nov.

19:00 → 20:30.

covers 7.1 → 9.4.

Schedule conflicts – send name & student #.

Last time: $T: V \to V$

U, W T-invariant subspaces

$V = U \oplus W$ Then

$$M_B(T) = \left(\begin{array}{c|c} M_{B_1}(T) & 0 \\ \hline 0 & M_{B_2}(T) \end{array}\right)$$

$B = B_1 \cup B_2$ B_1 basis for U

B_2 basis for W

Generalise: $V = U_1 \oplus U_2 \oplus \cdots \oplus U_k$,

U_i all T-invariant, then

$$M_B(T) = \begin{pmatrix} M_{B_1}(T) & & & 0 \\ & M_{B_2}(T) & & \\ & & \ddots & \\ 0 & & & M_{B_k}(T) \end{pmatrix}$$

Suppose U is T-invariant, and $\dim(U) = 1$. Say $U = \operatorname{span}\{v\}$.

We have $T(av) \in U$, so $T(av) = bv$

$$= \frac{b}{a}(av)$$

That is, av is an eigenvector of T; U is an eigenspace.

If $\dim(U_i) = 1$ for all i, where $V = U_1 \oplus U_2 \oplus \cdots \oplus U_k$, then V has a basis of eigenvectors, and $M_B(T)$ is diagonalisable.

If c_T, the characteristic poly of T has all distinct real roots, then V has a basis of eigenvectors.

Theorem Let $T: V \to V$ have char. poly $c_T(y) = (y-\lambda_1)^{m_1} \cdots (y-\lambda_k)^{m_k}$, where $\lambda_1, \dots, \lambda_k$ are real numbers. Then

$$V = G_{\lambda_1} \oplus \dots \oplus G_{\lambda_k},$$

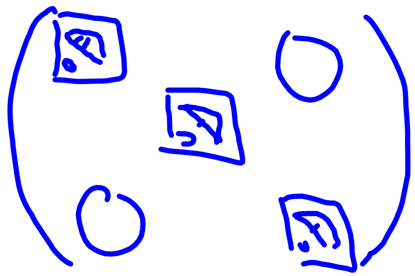
where

$$G_{\lambda_i} = \ker(\lambda_i I - T)^{m_i}.$$

The G_{λ_i} are T-invariant and there is a basis B_i of G_{λ_i} s.t. $M_{B_i}(T)$ is upper triangular.

Then $M_B(T)$ ~~is diagon~~ has diagonal block form, with the blocks upper triangular.

G_{λ_i} is a *generalized eigenspace*.



Example: Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ given by

$$T(a+bx+cx^2+dx^3) = (-a-b-c) + (3a+2b+3c-d)x + (2a+b+3c-d)x^2 + (2a+b+4c-2d)x^3$$

E = std basis for $\mathbb{P}_3$

$$M_E(T) = \begin{pmatrix} -1 & -1 & -1 & 0 \\ 3 & 2 & 3 & -1 \\ 2 & 1 & 3 & -1 \\ 2 & 1 & 4 & -2 \end{pmatrix}$$

$$c_T(y) = \det(yI - M_E)$$
$$= (y-1)^3(y+1).$$

eigenvalues are 1, with mult 3

-1, with mult 1.

$$G_{-1} = \ker(-I - T) = \{X \in \mathbb{R}^4 : (-I-T)X = 0\}$$
$$= \operatorname{span}\left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

9

$G_1(T) = \ker (I-T)^3$ $\boxed{T = m_E(T)!}$

1) Find a basis for $\ker(I-T)$

$$= \{ X : (I-T)X = 0 \}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

2) Find a basis for $\ker(I-T)^2$

i.e. solve $(I - m_E)^2 X = 0$

a basis is: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

3) Find a basis for $\ker(I-T)^3$

solve $(I - m_E)^3 X = 0$

a basis is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

Then T has block diagonal form ~~wrt~~

wrt the basis

$\{ 3 - x + x^2 + 9x^3,\ -1 + x + x^2 + x^3,\ x^2,\ x^2 + x^3 \}.$

Title: Oct 30 - 1:07 PM (6 of 6)