Lecture 22: Block diagonal form

Ex from last time:

$$T: \mathbb{P}_2 \longrightarrow \mathbb{P}_2$$

U_1, U_2 invariant subspaces

$U_1 = \text{span}\{x^2\}$, $U_2 = \text{span}\{1+x, x+x^2\}$

$B = \{1+x,\ x+x^2,\ x^2\}$

$$M_B(T) = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

We have $\mathbb{P}_2 = U_1 + U_2$, and also $U_1 \cap U_2 = \{0\}$.

<u>Defn</u>. If V has subspaces U, W st. $U \cap W = \{0\}$, $U + W = V$ then we say V is the <u>direct sum</u> of U and W, and write $V = U \oplus W$. We say that U, W are <u>complements</u> of each other.

$U_1 = \operatorname{span}\{x^2\}$ is T-invariant; in fact

$T(cx^2) = 5cx^2$

Thus x^2 is an eigenvector, with eigenvalue 5.

<u>Defn</u> Let $T: V \to V$ be a linear operator. If there is a vector $v \in V$ $v \neq 0$ and a scalar λ such that $T(v) = \lambda v$ then v is an <u>eigenvector</u> of T and λ is an <u>eigenvalue</u>. The <u>eigenspace</u> associated to λ is

$$E_\lambda = \{ v \in V : T(v) = \lambda v \}.$$

In particular, E_λ is a T-invariant subspace.

Check: let $v \in E_\lambda$; show $T(v) \in E_\lambda$.

$$T(T(v)) = T(\lambda v) \quad \text{as } v \in E_\lambda.$$

$$= \lambda T(v), \quad \text{as } T \text{ is linear.}$$

Hence $T(v) \in E_\lambda$.

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Theorem Let V be a vector space.

1) Let $\{e_1, \dots, e_n\}$ be a basis for V, $1 < k < n$. Let $U = \operatorname{span}\{e_1 \dots e_k\}$, $W = \operatorname{span}\{e_{k+1}, \dots, e_n\}$. Then $V = U \oplus W$.

2) Conversely, suppose $V = U \oplus W$, $U = \operatorname{span}\{e_1 \dots e_k\}$, $W = \{f_1, \dots, f_\ell\}$. Then $\{e_1 \dots e_k, f_1 \dots f_\ell\}$ is a basis for V.

Observe: if $V = U \oplus W$ and U, W are both T-invariant, then ${}_B M_B(T)$ has the block diagonal form

$$\begin{pmatrix} M_{B_1}(T) & 0 \\ 0 & M_{B_2}(T) \end{pmatrix}$$

where B_1 is a basis for U, B_2 is a basis for W and $B = B_1 \cup B_2$.

Ex. $V = M_{22}$.

A is symmetric if $A = A^T$

A is antisymmetric if $A = -A^T$.

$U = \{$ symmetric matrices in $M_{22} \}$.

$W = \{$ antisym. matrices $\}$.

U, W both subspaces.

Clear that $U \cap W = \{0\}$.

~~Stp~~ In fact: $U + W = M_{22}$

Consider $A + A^T$: $(A + A^T)^T = A^T + A$
$$= A + A^T$$

so $A + A^T \in U$.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A$$
$$= -(A - A^T)$$

Thus $M_{22} = U \oplus W$.

Consider $T: M_{22} \to M_{22}$
$$T(A) = A^T$$

Claim: U, W are both T-invariant.

Let $A \in U$. (symmetric).

Then $T(A) = A^T = A$.

So $T(A) \in U$. But even more,

T = identity on U.

Let $B \in W$. Then $T(B) = B^T = -B \in W$.

So W is T-invariant.

If B_1 is a basis for U, B_2 a basis for W, then $B = B_1 \cup B_2$

$$M_B(T) = \begin{pmatrix} M_{B_1}(T) & 0 \\ 0 & M_{B_2}(T) \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Find bases for U, W.

$$B_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$$B_2 = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

$$M_B(T) = \begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ & & & -1 \end{pmatrix}$$