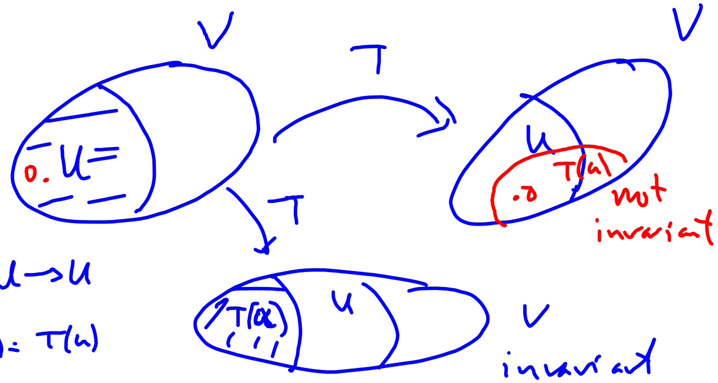
# Lecture 21: Invariant Subspaces.

**Defn.** $U$ subspace of $V$, $T: V \to V$.

$U$ is <u>invariant</u> if $T(U) \subseteq U$.




$T|_U : U \to U$

$T|_U(u) = T(u)$

**Theorem** Let $T: V \to V$ be a linear operator, $U$ a $T$-invariant subspace of $V$. Let $B_1 = \{e_1 \dots e_k\}$ be any basis of $U$, and $B = \{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$ an extension of $B_1$ to a basis of $V$. Then $M_B(T)$ has the <u>block triangular</u> form:

$$M_B(T) = \left(\begin{array}{c:c} M_{B_1}(T|_U) & Y \\ \hdashline O & Z \end{array}\right)$$

where $M_{B_1}(T|_U)$ is the matrix of $T$ as a linear operator on $U$ w.r.t. the basis $B_1$.

Proof. $M_{B_1}(T) = \left( c_{B_1} T(e_1) \;\; c_{B_1} T e_2 \; \ldots \; c_{B_1} T e_k \right)$

$M_B(T) = \left( c_B T(e_1) \; \ldots \; c_B T e_n \right)$

$e_1 \in U$, so $T(e_1) \in U$

$T(e_1) = r_1 e_1 + r_2 e_2 + \ldots + r_k e_k + 0 e_{k+1} \ldots + 0 e_n$

$$c_B(T(e_1)) = \begin{pmatrix} r_1 \\ \vdots \\ r_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} c_{B_1}(T(e_1)) \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$ as required.

Ex. $T : \mathbb{P}_2 \to \mathbb{P}_2$

$$T(a + bx + cx^2) = 3a - b + (4a + b)x + (6a - 3b + 5c)x^2$$

$E = \{1, x, x^2\}$.

$$M_E(T) = \left( c_E T(1) \;\; c_E(T(x)) \;\; c_E T(x^2) \right)$$
$$= \begin{pmatrix} 3 & -1 & 0 \\ 4 & 1 & 0 \\ 6 & -3 & 5 \end{pmatrix}$$

Observe: $U_1 = \text{span}\{x^2\}$ is invariant.

Why? $T(cx^2) = 0 + 0x + 5cx^2 \in U_1$.

But also: $U_2 = \text{span}\{1+x,\ x+x^2\}$ is also $T$-invariant.

$$T(1+x) = (3-1) + (4+1)x + (6-3+0)x^2$$
$$= 2 + 5x + \cancel{6}^{\,3}x^2$$
$$= 2(1+x) + 3(x+x^2) \in U_2$$

$$T(x+x^2) = -1 + x + (-3+5)x^2$$
$$= -1 + x + 2x^2$$
$$= -1(1+x) + 2(x+x^2) \in U_2.$$

If $u \in U_2$, $u = \cancel{a_1}^{\,r_1}(1+x) + \cancel{a_2}^{\,r_2}(x+x^2)$

so $T(u) = \cancel{a_1}^{\,r_1}\,T(1+x) + \cancel{a_2}^{\,r_2}\,T(x+x^2) \in U_2$

$U_2 = \operatorname{span}\{1+x,\ x+x^2\}$.

Expand to a basis $\mathcal{B}$ of $\mathbb{P}_2$:

$$\mathcal{B} = \{1+x,\ x+x^2,\ x^2\} = \{e_1, e_2, e_3\}.$$

$$M_{\mathcal{B}}(T) = \begin{pmatrix} c_{\mathcal{B}}T(e_1) & c_{\mathcal{B}}T(e_2) & c_{\mathcal{B}}(T(e_3)) \end{pmatrix}$$

$$= \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 3 & 2 & 0 \\ \hline 0 & 0 & 5 \end{array}\right)$$

Find characteristic polynomial of $T$.

$$c_T(y) = \det\left(yI - M_{\mathcal{B}}(T)\right)$$

$$= \det\begin{pmatrix} y-2 & 1 & 0 \\ -3 & y-2 & 0 \\ 0 & 0 & y-5 \end{pmatrix}$$

$$= \det\begin{pmatrix} y-2 & 1 \\ -3 & y-2 \end{pmatrix}(y-5)$$

$$= \left[(y-2)(y-2) - (-3)\right](y-5)$$

$$c_T(y) = (y^2 - 4y + 7)(y-5).$$