Lecture 20

$B = \{e_1 \dots e_n\}$

$T : V \longrightarrow W$

B, D ordered bases for V, W respectively

$$M_{DB}(T) = \left(c_D(T(e_1)) \;\; c_D(T(e_2)) \;\dots\; c_D(T(e_n)) \right)$$

$$T(v) = c_D^{-1}\left(M_{DB}\, c_B(v)\right)$$

If $T = Id : V \longrightarrow V$, get change-of-basis matrix:

$$M_{DB}(Id) = P_{D \leftarrow B} = \left(c_D(e_1) \dots c_D(e_n) \right)$$

$$M_B(T) = M_{BB}(T)$$

$$M_B(T) = P_{D \leftarrow B}^{-1}\, M_D(T)\, P_{D \leftarrow B}$$

Theorem 1) Let $T: V \to V$ be a linear transformation, $\mathcal{B}, \mathcal{D}$ ordered bases of V. Then the matrices $M_{\mathcal{B}}(T)$, $M_{\mathcal{D}}(T)$ are similar.

2) Let A, A' be similar matrices, with $A' = P^{-1}AP$. Then there is a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ and a basis B of $\mathbb{R}^n$ s.t. $A = M_E(T)$ and $A' = M_B(T)$. In fact, $T = T_A$ ($T_A(X) = AX$) and B is the set of columns of P.

(E is the standard basis).

Defn. Let $T: V \to V$ be a lin. trans, M_B any matrix representing T.

1) $\det(T) = \det(M_B)$

2) characteristic polynomial of T is the char. poly of M_B; i.e. $\det(tI - M_B)$.

Recall: A matrix, solve $Ax = \lambda x$

$$Ax - \lambda I x = 0$$

$$(A - \lambda I)x = 0$$

non-trivial soln iff $\underbrace{\det(A - \lambda I)}_{\text{polynomial in } t.} = 0$

char poly is $\det(A - xI)$

Check well-defined: let D be another basis, M_D another matrix representing T.

Then $M_D = P^{-1} M_B P$, $\left(P = P_{B \leftarrow D} \right)$.

$$\det(tI - M_D) = \det(tI - P^{-1}M_B P)$$
$$= \det(tP^{-1}IP - P^{-1}M_B P)$$
$$= \det\left(P^{-1}(tI - M_B)P\right)$$
$$= \det(P^{-1}) \det(tI - M_B) \det(P)$$
$$= \det(tI - M_B).$$

Ex $T : \mathbb{P}_2 \to \mathbb{P}_2$

$T(1) = x^2 - 1$ $\qquad e_1 = 1$

$T(x) = 3x$ $\qquad e_2 = x$

$T(x^2) = x^2 + 2$ $\qquad e_3 = x^2$

$$M_E(T) = \left(c_E T(e_1) \quad c_E(T(e_2)) \quad c_E(T(e_3)) \right)$$
$$= \begin{pmatrix} -1 & 0 & 2 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$\cancel{B} D = \{ f_1, f_2, f_3 \}$, $f_1 = 1$, $f_2 = x^2$,

$f_3 = x$

$$M_D(T) = \big(c_D T(f_1) \quad c_D T(f_2) \quad c_D T(f_3) \big)$$

$$= \big(c_D(x^2-1) \quad c_D(x^2+2) \quad c_D(3x) \big)$$

$$= \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 3 \end{array}\right)$$

Let U_1 = subspace of even polys in P_2

U_2 = — — odd — —

basis for U_1 is $\{1, x^2\}$, basis for U_2 is $\{x\}$.

$D = \{$ basis for U_1, basis for $U_2 \}$.

<u>Definition</u> Let $T: V \to V$ be a linear operator, U a subspace of V. Write $T(U) = \{T(u) : u \in U\}$. If $T(U) \subseteq U$ then U is said to be <u>T-invariant</u>.

In this case, the <u>restriction</u> of T to U is a linear operator on U.

<u>Ex</u>. In above. $T: U_1 \to U_1$,
$T: U_2 \to U_2$.

U_1, U_2 are both T-invariant.