Math 2R Lecture 2

Abstract Vector Space

<u>Defn</u> A vector space over $\mathbb{R}$ is a set V (elements are called vectors) with two operations vector addition and scalar mult^n, which satisfy the following:

A1 $u, v \in V \Rightarrow u+v \in V$

A2 $u+v = v+u$

A3 $u+(v+w) = (u+v)+w$

A4 there is $\bar{0} \in V$ such that for all $v \in V$, $\bar{0}+v = v+\bar{0} = v$.

A5 for every $v \in V$, there is $-v \in V$ such that $v+(-v) = \bar{0}$

S1 $a \in \mathbb{R},\ v \in V \Rightarrow av \in V$

S2 $a(v+w) = av + aw$

S3 $(a+b)v = av + bv$

S4 $a(bv) = (ab)v$

S5 $1v = v$

<u>Examples</u> (0) $\mathbb{R}^n = V$

$\mathbb{R}^n = \{(x_1, x_2, \dots x_n) : x_i \in \mathbb{R}\}$

1) $V = M_{mn}$ = set of $m \times n$ matrices over $\mathbb{R}$

vector add^n is matrix addition
scalar mult^n is mult^n of matrices by real numbers

$A = (a_{ij})$, $rA = (ra_{ij})$.

2) $V = \mathbb{P}_n$ = polynomials of degree $\leq n$

A polynomial is a sum of powers of x, with coefficients.

degree of $p(x) = a_0 + a_1 x + \dots + a_n x^n$

is n, where $a_n \neq 0$.

$$\begin{array}{l} a_0 + a_1 x + \dots + a_n x^n \\ b_0 + b_1 x + \dots + b_n x^n \\ \hline a_0 + b_0 + (a_1 + b_1) x + \dots + (a_n + b_n) x^n \end{array}$$

$$c(a_0 + \dots + a_n x^n) = ca_0 + ca_1 x + \dots + ca_n x^n$$

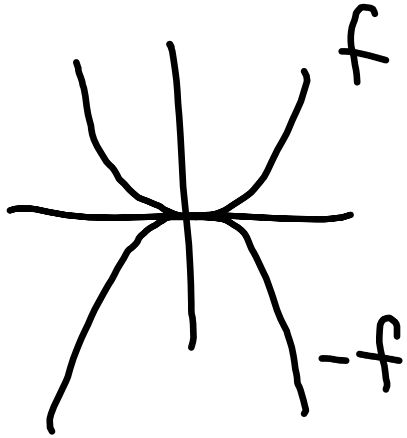
2)' $\mathbb{P}$ = set of all polynomials.

3) $\mathbb{F}[a,b]$ = set of all functions with domain $[a,b]$

$(f+g)(x) = f(x) + g(x)$

$\bar{0}$ function is defined by $\bar{0}(x) = 0$.

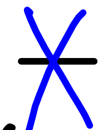
$(cf)(x) = cf(x)$



$\mathbb{C}[a,b]$ = set of continuous functions on $[a,b]$

f cont^{d} at c if $\lim_{x \to c} f(x) = f(c)$ ✓

$\lim_{x \to c} (f(x)+g(x)) = \lim f + \lim g$



$\mathbb{D}[a,b]$ = set of differentiable functions

Properties of vector addition and scalar multiplication that follow from the axioms.

Proposition: There is a unique solution to the equation $x+v=w$, for any $v, w \in V$.

Proof Assume $x + v = w$

Then $(x+v) + (-v) = w + (-v)$ A5

$x + (v + (-v)) = w + (-v)$ A3

$x + \bar{0} = w + (-v)$ A5

$x = w + (-v)$ A4

This shows existence. Then verify uniqueness.

Prop The zero vector is unique.

Proof. Suppose there is another vector $\bar{0}^*$, which also ~~sat~~ has the properties given in A4. i.e.

for all $v \in V$ $\quad v + \bar{0}^* = \bar{0}^* + v = v.$

Let $v = \bar{0}$. Then $\bar{0} + 0^* = \bar{0}$.

But also, $\bar{0}$ satisfies A4. Take $v = \bar{0}^*$.

Then $\quad \bar{0} + \bar{0}^* = \bar{0}^*$

Hence $\quad \bar{0}^* = \bar{0}$.