Lecture 19: Change of Basis

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "C_B"'] & W \arrow[d, "C_D"] \\ \mathbb{R}^n \arrow[r, dashed, "T_{DB}"'] & \mathbb{R}^m \end{tikzcd}$$

$$T_{DB} = C_D \, T \, C_B^{-1}$$

$M_{DB}(T)$ is the matrix which represents T_{DB}.

$$T_{DB}(X) = M_{DB} X.$$

$$M_{DB} = \big(C_D T(e_1) \quad C_D T(e_2) \ \ldots \ C_D T(e_n) \big)$$

$$T = C_D^{-1} \, T_{DB} \, C_B$$

For any vector $\bar{v} \in V$,

$$T(\bar{v}) = C_D^{-1}\big(T_{DB}(C_B(\bar{v}))\big)$$

$$C_D(T(v)) = T_{DB}(C_B(\bar{v}))$$

$$C_D(T(\bar{v})) = M_{DB} \, C_B(\bar{v}) \qquad (*)$$

Consider $T = Id : V \to V$.

~~Then~~ $Id(\bar{v}) = \bar{v}$.

$(*)$ says: $C_D(\bar{v}) = M_{DB} \, C_B(\bar{v})$

Defn $M_{\mathcal{D}\mathcal{B}}(Id) = P_{\mathcal{D}\leftarrow\mathcal{B}}$ is called the change of basis matrix, and satisfies $c_{\mathcal{D}}(v) = P_{\mathcal{D}\leftarrow\mathcal{B}}\, c_{\mathcal{B}}(v)$

Ex. Let $V = \mathbb{P}_2$ $\quad \mathcal{B} = \{\underset{e_1}{\underset{\shortparallel}{1}}, \underset{e_2}{\underset{\shortparallel}{x}}, \underset{e_3}{\underset{\shortparallel}{x^2}}\}$

$\mathcal{D} = \{1, x-1, (x-1)^2\}$

$P_{\mathcal{D}\leftarrow\mathcal{B}} = \left(c_{\mathcal{D}}(e_1) \;\; c_{\mathcal{D}}(e_2) \;\; c_{\mathcal{D}}(e_3) \right)$

$e_1 = 1 = 1 \qquad c_{\mathcal{D}}(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$e_2 = x = (x-1) + 1 \qquad c_{\mathcal{D}}(e_2) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$e_3 = x^2 = (x-1)^2 + 2(x-1) + 1$

$c_{\mathcal{D}}(e_3) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$P_{\mathcal{D}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consider $a + bx + cx^2$

$$C_D(a+bx+cx^2) = P_{D \leftarrow B} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} a+b+c \\ b+2c \\ c \end{pmatrix}$$

$$C_D^{-1} C_D(a+bx+cx^2) = (a+b+c)1 + (b+2c)(x-1) + c(x-1)^2$$

Given $T: V \to V$, B, D two bases for V. We have:

$$c_D(v) = P_{D \leftarrow B}\, c_B(v)$$

$$c_B(T(v)) = M_B(T)\, c_B(v)$$

$$c_D(T(v)) = M_D(T)\, c_D(v)$$

How do M_B and M_D compare?

$$P_{D \leftarrow B}\, c_B(T(v)) = c_D(T(v))$$

$$= M_D(T)\, c_D(\cancel{T(v)}\, v)$$

$$= M_D(T)\, P_{D \leftarrow B}\, c_B(\bar{v})$$

$$\cancel{c_D(T(v))}$$

$$\| $$

$$P_{D \leftarrow B}\, M_B(T)\, c_B(v)$$

Since this holds for all vectors $\bar{v}$,

then $$P_{D \leftarrow B}\, M_B = M_D\, P_{D \leftarrow B}$$

i.e. $$M_B = P_{D \leftarrow B}^{-1}\, M_D\, P_{D \leftarrow B}$$

M_B and M_D are <u>similar</u> matrices.