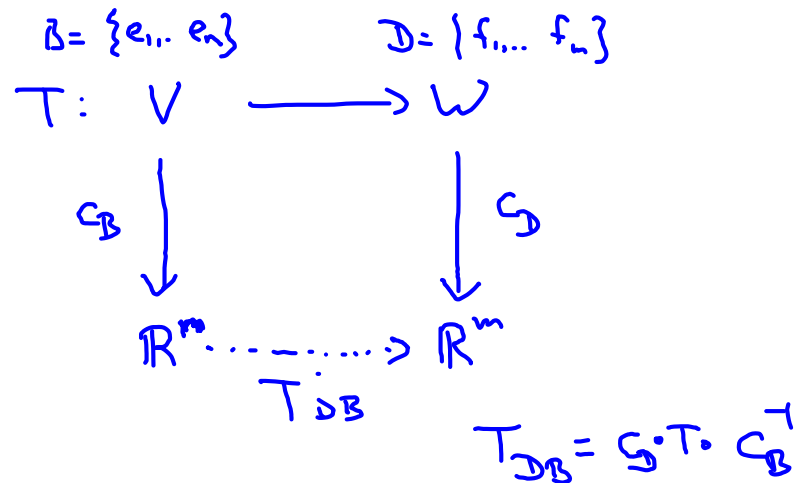


# Lecture 18: More on matrix of a linear transformation.



$M_{DB}$  is the matrix of  $T_{DB}$

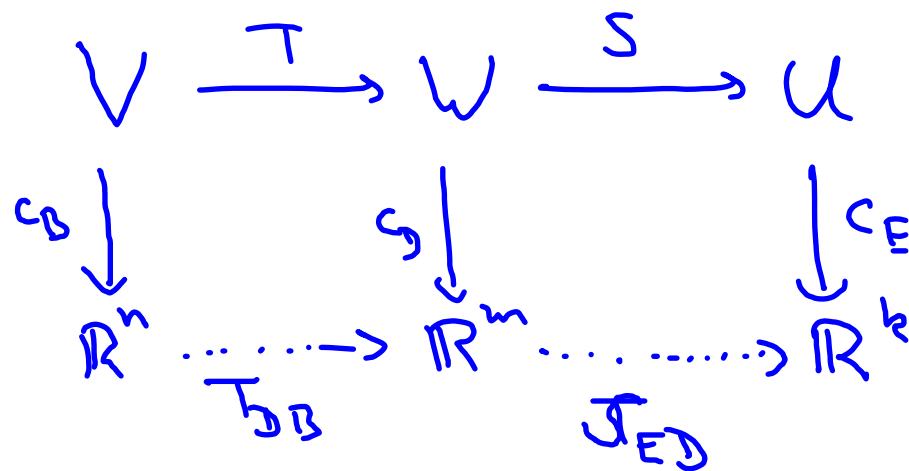
$$M_{DB} = \left( S T C_B^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, S T C_B^{-1} \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \right)$$

$$M_{DB} = (S T(e_1) \dots S T(e_n))$$

To find  $M_{DB}$ :

- 1) calculate  $T(e_i)$  for  $i=1 \dots n$
- 2) write  $T(e_i)$  in the basis  $D$
- 3) take the coeffs of  $T(e_i)$  as columns

Composition of linear transformations.



$$S \circ T: V \rightarrow U$$

$$S_{ED} \circ T_{DB}: \mathbb{R}^n \rightarrow \mathbb{R}^k$$

$$M_{EB}(S \circ T) = \text{matrix of } S_{ED} \circ T_{DB}$$

$$= M_{ED}(S) M_{DB}(T)$$

$$\begin{array}{c}
 \underbrace{k \times m} \quad \underbrace{m \times n} \\
 \underbrace{\hspace{10em}} \\
 k \times n
 \end{array}$$

$$\text{Ex. } T: \mathbb{P}_3 \rightarrow M_{22}$$

$$T(p(x)) = \begin{pmatrix} p(0) & p(1) \\ p(2) & p(3) \end{pmatrix}$$

$$B = \left\{ \begin{matrix} e_1 & e_2 & e_3 & e_4 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x & x^2 & x^3 \end{matrix} \right\}$$

$$D = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\text{Want } M_{D,B}(T) = (c_D T(e_1) \dots c_D T(e_4))$$

$$T(e_1) = T(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$c_D(Te_1) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$T(e_1) = T(x) = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$$

$$\hookrightarrow T(e_1) = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$T(e_2) = T(x^2) = \begin{pmatrix} 0 & 1 \\ 4 & 9 \end{pmatrix}$$

$$\hookrightarrow T(e_2) = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \end{pmatrix}$$

$$T(e_3) = T(x^3) = \begin{pmatrix} 0 & 1 \\ 8 & 27 \end{pmatrix}$$

$$\hookrightarrow T(e_3) = \begin{pmatrix} 0 \\ 1 \\ 8 \\ 27 \end{pmatrix}$$

$$M_{DB}(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix}$$

$$T(a+bx+cx^2+dx^3) = S^{-1} M_{DB} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$= S^{-1} \begin{pmatrix} a \\ a+b+c+d \\ a+2b+4c+8d \\ a+3b+9c+27d \end{pmatrix}$$

$$T(a+bx+cx^2+dx^3) = \begin{pmatrix} a & a+b+c+d \\ a+2b+4c+8d & a+3b+9c+27d \end{pmatrix}$$

$$T: \mathbb{P}_3 \rightarrow M_{2,2} \quad T(p(x)) = \begin{pmatrix} p(0) & p(1) \\ p(2) & p(3) \end{pmatrix}$$

$$B' = \{1, x, x^2, x^3 + x^2 + x + 1\}$$

$$D' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Find  $M_{D', B'}$ .

$$T(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$c_{D'}(T(1)) = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$T(x) = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = a_1 \begin{pmatrix} \phantom{0} \\ \phantom{0} \end{pmatrix} + a_2 \begin{pmatrix} \phantom{0} \\ \phantom{0} \end{pmatrix} \dots$$

$$= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

$$T(x^2)$$

$$T(x^3 + x^2 + x + 1) = \begin{pmatrix} 1 & 4 \\ 15 & 40 \end{pmatrix}$$

$$c_{D'}(T(x^3 + x^2 + x + 1)) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

$$M_{D', B'} = \begin{pmatrix} 2 & a_1 & b_1 & c_1 \\ -1 & a_2 & \cdot & \cdot \\ 1 & a_3 & \cdot & \cdot \\ 1 & a_4 & \cdot & c_4 \end{pmatrix}$$

Fix  $T: V \rightarrow W$ .

Choose  $B = \{e_1, \dots, e_r, e_{r+1}, \dots, e_n\}$  so that  $\{e_{r+1}, \dots, e_n\}$  are a basis for  $\ker(T)$ .

Then  $\{T(e_1), \dots, T(e_r)\}$  is a basis for  $\text{Im}(T)$ . Extend to a basis

$D = \{T(e_1), \dots, T(e_r), f_{r+1}, \dots, f_m\}$  for  $W$

$$M_{DB}(T) = ({}_D T(e_1) \dots {}_D T(e_r) \dots {}_D T(e_n))$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Given  $T: V \rightarrow V$  can I choose  
a basis  $B$  for  $V$  so that  $M_{BB}(T)$   
is as simple as possible?

Defn A lin-trans. from  $V \rightarrow V$   
is called a linear operator. We will  
write  $M_{BB}(T) = M_B(T)$ .