Lecture 18: More on matrix of a linear transformation.

$B = \{e_1, \dots, e_n\}$ $\qquad$ $D = \{f_1, \dots, f_m\}$

$$\begin{tikzcd}
T: V \arrow[r] \arrow[d, "c_B"'] & W \arrow[d, "c_D"] \\
\mathbb{R}^n \arrow[r, dotted, "T_{DB}"'] & \mathbb{R}^m
\end{tikzcd}$$

$$T_{DB} = c_D \circ T \circ c_B^{-1}$$

M_{DB} is the matrix of T_{DB}

$$M_{DB} = \left(c_D T c_B^{-1}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \; \dots \; c_D T c_B^{-1}\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix} \right)$$

$$M_{DB} = \left(c_D T(e_1) \; \dots \; c_D T(e_n) \right)$$

To find M_{DB}:

1) calculate $T(e_i)$ for $i = 1..n$
2) write $T(e_i)$ in the basis D
3) take the coeffs of $T(e_i)$ as columns

Composition of linear transformations.

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "c_B"'] & W \arrow[r, "S"] \arrow[d, "c_D"'] & U \arrow[d, "c_E"] \\
\mathbb{R}^n \arrow[r, dotted, "T_{DB}"'] & \mathbb{R}^m \arrow[r, dotted, "S_{ED}"'] & \mathbb{R}^k
\end{tikzcd}$$

$S \circ T : V \to \cancel{W}\ U$

$S_{ED} \circ T_{DB} : \mathbb{R}^n \to \mathbb{R}^k$

$M_{EB}(S \circ T) =$ matrix of $S_{ED} \circ T_{DB}$

$$= \underbrace{\underset{\substack{\cancel{m \times k} \\ k \times m}}{M_{ED}(S)} \; \underset{m \times n}{M_{DB}(T)}}_{k \times n}$$

Ex. $T : \mathbb{P}_3 \longrightarrow M_{22}$

$$T(p(x)) = \begin{pmatrix} p(0) & p(1) \\ p(2) & p(3) \end{pmatrix}$$

$$B = \{ \underset{e_1}{1}, \underset{e_2}{x}, \underset{e_3}{x^2}, \underset{e_4}{x^3} \}$$

$$D = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Want $M_{DB}(T) = \left(c_D T(e_1) \; \ldots \; c_D T(e_4) \right)$

$$T(e_1) = T(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$c_D(Te_1) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$T(e_2) = T(x) = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$$

$$S_D\, T(e_2) = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$$T(e_3) = T(x^2) = \begin{pmatrix} 0 & 1 \\ 4 & 9 \end{pmatrix}$$

$$S_D\, T(e_3) = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \end{pmatrix}$$

$$T(e_4) = T(x^3) = \begin{pmatrix} 0 & 1 \\ 8 & 27 \end{pmatrix}$$

$$S_D\, T(e_4) = \begin{pmatrix} 0 \\ 1 \\ 8 \\ 27 \end{pmatrix}$$

$$M_{DB}(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix}$$

$$T(a + bx + cx^2 + dx^3) = S_D^{-1} M_{DB} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$= S_D^{-1} \begin{pmatrix} a \\ a + b + c + d \\ a + 2b + 4c + 8d \\ a + 3b + 9c + 27d \end{pmatrix}$$

$$T(a + bx + cx^2 + dx^3) = \begin{pmatrix} a & a + b + c + d \\ a + 2b + 4c + 8d & a + 3b + 9c + 27d \end{pmatrix}$$

$T: \mathbb{P}_3 \to M_{22} \quad T(p(x)) = \begin{pmatrix} p(0) & p(1) \\ p(2) & p(3) \end{pmatrix}$

$B' = \{1, x, x^2, x^3 + x^2 + x + 1\}.$

$D' = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

Find $M_{D'B'}$.

$T(1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 1\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$C_{D'}(T(1)) = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$T(x) = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = a_1\begin{pmatrix} & \end{pmatrix} + a_2\begin{pmatrix} & \end{pmatrix} \ldots$

$= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$

$T(x^2)$

$T(x^3 + x^2 + x + 1) = \begin{pmatrix} 1 & 4 \\ 15 & 40 \end{pmatrix}$

$C_{D'}(T x^3 + \ldots) = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$

$M_{D'B'} = \begin{pmatrix} 2 & a_1 & b_1 & c_1 \\ 1 & a_2 & \cdot & \cdot \\ 1 & a_3 & \cdot & \vdots \\ 1 & a_4 & & c_4 \end{pmatrix}$

Fix $T: V \to W$.

Choose $\mathcal{B} = \{e_1 \dots e_r, e_{r+1} \dots e_n\}$ so that $\{e_{r+1} \dots e_n\}$ are a basis for $\mathrm{Ker}(T)$.

Then $\{T(e_1) \dots, T(e_r)\}$ is a basis for $\mathrm{Im}(T)$. Extend to a basis

$\mathcal{D} = \{T(e_1) \dots T(e_r), f_{r+1}, \dots f_m\}$ for W

$$M_{\mathcal{D}\mathcal{B}}(T) = \left(C_{\mathcal{D}} T(e_1) \dots C_{\mathcal{D}} T(e_r) \dots C_{\mathcal{D}} T(e_n) \right)$$

$$= r \rightarrow \begin{pmatrix} 1 & 0 & & 0 & 0 & 0 \\ 0 & 1 & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & 1 & \vdots & \dots \\ 0 & 0 & & 0 & 0 & 0 \\ & & & \vdots & & \end{pmatrix}$$

$$= \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Given $T: V \to V$ can I choose a basis $\mathcal{B}$ for V so that $M_{\mathcal{B}\mathcal{B}}(T)$ is as simple as possible?

<u>Defn</u> A lin. trans. from $V \to V$ is called a <u>linear operator</u>. We will write $M_{\mathcal{B}\mathcal{B}}(T) = M_{\mathcal{B}}(T)$.