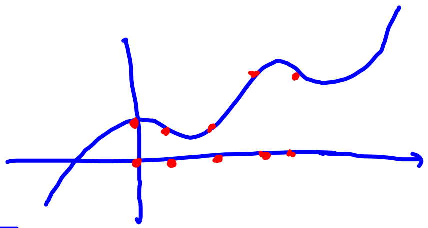
Lecture 17 Linear Transformations as Matrices

Ex. $T: \mathbb{P}_n \longrightarrow \mathbb{R}^{n+1}$

$$T(p(x)) = (p(0), p(1), p(2), \ldots, p(n))$$

Claim: T is invertible.



By the theorem, it suffices to show that ~~Ez~~ T is 1-1 and onto.

Let $p(x) \in \mathrm{Ker}(T)$. Then

$$T(p(x)) = (p(0), \ldots p(n)) = (0, 0, \ldots, 0)$$

$p(x)$ is a polynomial of degree at most n.

$p(x)$ has $n+1$ zeroes.

This can only happen if $p(x)$ is the zero polynomial.

Thus $\mathrm{Ker}(T) = \{\bar{0}\}$, so T is 1-1.

$\mathrm{im}(T) \subseteq \mathbb{R}^{n+1}$, so

$\dim(\mathrm{im}(T)) \le \dim(\mathbb{R}^{n+1}) = n+1.$

Dimension thm: $\dim(\mathbb{P}^n) = \dim(\mathrm{im}(T)) + \dim(\ker(T))$

$n+1 = \dim(\mathrm{im}(T)) + 0.$

Thus $\dim(\mathrm{im}(T)) = \dim(\mathbb{R}^{n+1})$, so

$\mathrm{im}(T) = \mathbb{R}^{n+1}$

That is, T is onto.

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear trans,

then there is a matrix $A_T^{m\times n}$ such that

for any $X \in \mathbb{R}^n$, $T(X) = A_T X$

$A_T = (T(e_1) \ldots T(e_n))$.

Generalise this to $T: V \to W$, for any

vector spaces V, W.

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Let V have dimension n.

Fix an ordered basis $\mathcal{B} = \{e_1, \dots, e_n\}$.

<u>Defn</u> The <u>coordinate</u> <u>transformation</u>

$c_{\mathcal{B}} : V \to \mathbb{R}^n$, given by

$$c_{\mathcal{B}}(v) = c_{\mathcal{B}}(v_1 e_1 + v_2 e_2 + \dots + v_n e_n)$$

$$= \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

<u>Thm</u> $c_{\mathcal{B}}$ is an isomorphism.

Ex. $V = M_{22}$ $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$\qquad e_1 \qquad e_2 \qquad e_3 \qquad e_4$

$$C_B\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = C_B\left(ae_1 + be_2 + ce_3 + de_4\right) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$D = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

$\qquad f_1 \qquad f_2 \qquad f_3 \qquad f_4$

$$C_D\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = C_D\left(\tfrac{1}{2}(a+d) f_1 + \tfrac{1}{2}(b+c) f_2 + \tfrac{1}{2}(a-d) f_3 + \tfrac{1}{2}(c-b) f_4\right)$$

$$= \begin{pmatrix} \frac{1}{2}(a+d) \\ \frac{1}{2}(b+c) \\ \frac{1}{2}(a-d) \\ \frac{1}{2}(c-b) \end{pmatrix}$$

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Let $T: V \to W$ lin. trans.

$B = \{e_1 \dots e_n\}$ ordered basis for V

$D = \{f_1 \dots f_m\}$ ordered basis for W

We have $\quad c_B : V \to \mathbb{R}^n$

$\qquad\qquad\quad c_D : W \to \mathbb{R}^m$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "c_B"'] & W \arrow[d, "c_D"] \\ \mathbb{R}^n \arrow[r, dashed, "T_{DB}"'] & \mathbb{R}^m \end{tikzcd}$$

For $X \in \mathbb{R}^n$, $\quad c_D T c_B^{-1}(X) \in \mathbb{R}^m$

Define $\quad T_{DB} = c_D \circ T \circ c_B^{-1}$

T_{DB} can be written as a matrix multiplication; we'll call that matrix $M_{DB}(T)$.

Columns of $M_{DB}(T)$ are

$c_D T c_B^{-1}$ (standard basic vector for $\mathbb{R}^n$)

$c_D T(e_i)$

$$M_{DB}(T) = \left(c_D T(e_1) \; c_D T(e_2) \; \dots \; c_D T(e_n) \right)$$