Lecture 16: Isomorphisms and Inverses

Proof of Corollary:

"$\Longrightarrow$" Suppose V, W are isomorphic.

Let $T: V \to W$ be the iso^m. ~~and~~

~~…~~ $\ker(T) = \{0\}$, so $\dim(\ker(T)) = 0$.

By dimension theorem, $\dim(V) = \dim(\operatorname{im}(T))$.

Since T is onto, $\operatorname{im}(T) = W$; thus

$\dim(\operatorname{im}(T)) = \dim(W)$. i.e. $\dim(V) = \dim(W)$.

"$\Longleftarrow$" Suppose $\dim(V) = \dim(W)$.

Fix a basis $\{e_1, \dots, e_n\}$ for V, and a basis $\{f_1, \dots, f_n\}$ for W. Define $T: V \to W$ by $T(e_i) = f_i$. By 3) $\Rightarrow$ 1) of the theorem, T is an isomorphism $\blacksquare$

Proof of theorem.

1) $\Rightarrow$ 2). Assume T is an isomorphism.

Let $\{e_1, \dots, e_n\}$ be any basis for V.

Show $\{T(e_1), \dots, T(e_n)\}$ is a basis for W.

Suppose $a_1 T(e_1) + \dots + a_n T(e_n) = 0$

$$T(a_1 e_1 + \dots + a_n e_n) = 0$$

Thus $a_1 e_1 + \dots + a_n e_n = 0$, ~~as~~ as T is 1-1.

Hence $a_1 = a_2 = \dots = a_n = 0$, as $\{e_1, \dots, e_n\}$ is independent. Thus $\{T(\cancel{a_1}\, e_1), \dots, T(\cancel{a_n}\, e_n)\}$ is independent.

To show $\{T(e_1), \ldots, T(e_n)\}$ spans W,
let $w \in W$. Since T is onto, there
is $v \in V$ st. $w = T(v)$.

$v = r_1 e_1 + \cdots + r_n e_n$ for some $r_1, \ldots, r_n \in \mathbb{R}$,

so
$$\begin{aligned} w &= T(r_1 e_1 + \cdots + r_n e_n) \\ &= r_1 T(e_1) + \cdots + r_n T(e_n) \\ &\in \operatorname{span}\{T(e_1), \ldots, T(e_n)\}. \end{aligned}$$

2) $\Rightarrow$ 3) immediate, since V has some basis.

3) $\Rightarrow$ 1) Assume there is some basis $\{e_1, \ldots, e_n\}$ of V st. $\{T(e_1), \ldots, T(e_n)\}$ is a basis for W. Let $w \in W$, so

$$\begin{aligned} w &= a_1 T(e_1) + \cdots + a_n T(e_n) \\ &= T(a_1 e_1 + \cdots + a_n e_n) \end{aligned}$$

So $w \in \operatorname{im}(T)$. Thus $\operatorname{im}(T) = W$, so
T is onto.

$$\underbrace{\dim(V)}_{= n} = \underbrace{\dim \ker(T)}_{=0} + \underbrace{\dim \operatorname{im}(T)}_{= n}$$

Thus $\dim(\ker(T)) = 0$, so T is 1-1.

Composition of linear transformations:

$T: V \to W$, $S: W \to U$

then $S \circ T: V \to U$

$$S \circ T(v) = S(T(v))$$

Observation: $S \circ T$ is also a linear transformation.

We say that $T: V \to W$ is <u>invertible</u> if there is $S: W \to V$ s.t.

for all $v \in V$, $S(T(v)) = v$ $\quad S \circ T = Id_V$

and for all $w \in W$, $T(S(w)) = w$ $\quad T \circ S = Id_W$

<u>Theorem</u> $T: V \to W$ a lin trans.

T is invertible if and only if T is an isomorphism.

Pf. Suppose T is invertible; S be the inverse.

Let $v \in V$ st. $T(v) = 0_W$

Then $$\underset{\substack{\| \\ S\circ T(v)}}{S(T(v))} = S(0_W) = 0_V$$

But $S \circ T = Id_V$, so $S \circ T(v) = v$.

i.e. $v = 0_V$. Thus $\ker(T) = \{0\}$ and T is one-to-one.

Let $w \in W$. Now $T(S(w)) = w$, as $T \circ S = Id_W$. Thus $w \in \mathrm{im}(T)$.

Thus T is onto.

Now suppose T is an isomorphism.

We need to show there is a function $S : W \to V$ st. $T \circ S = Id_W$, $S \circ T = Id_V$.

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Let $\{e_1, \dots, e_n\}$ be a basis for V, so that $\{T(e_1), \dots T(e_n)\}$ is a basis for W.

Define $S: W \to V$ by
$$S(T(e_i)) = e_i$$

Now $S \circ T$ acts as Id_V on the basis $\{e_1, \dots e_n\}$. By uniqueness statement, $S \circ T = Id_V$.

Now $T \circ S(T(e_i)) = T(S(T(e_i)))$
$$= T(e_i)$$

Again $T \circ S$ acts as Id_W on the basis, hence $T \circ S = Id_W$. $\blacksquare$

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