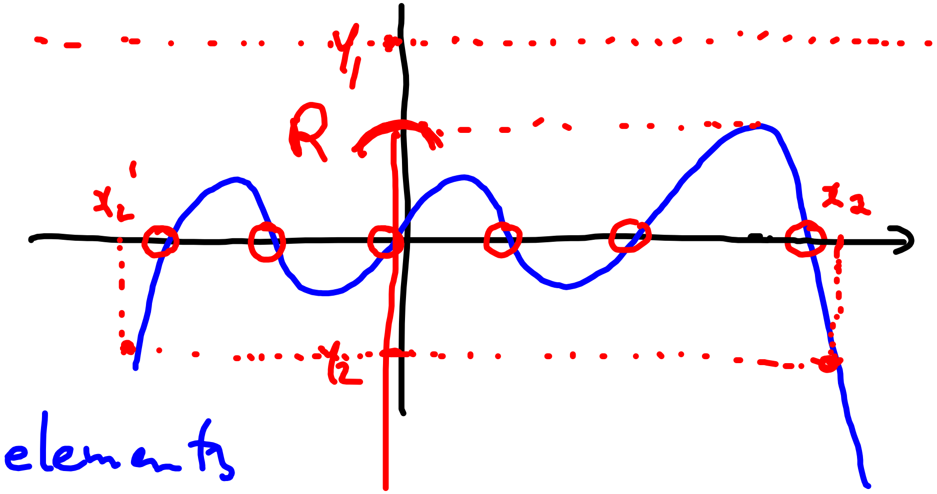
Lecture 15: Isomorphisms

$f : \mathbb{R} \to \mathbb{R}$



$\ker(f) = \{x \in \mathbb{R} : \underbrace{f(x) = 0}\}$

the kernel has 6 elements

to find the kernel, solve $f(x) = 0$

$\text{im}(f) = \{ y \in \mathbb{R} :$ there exists an x with $f(x) = y\}$

$y_1 \notin \text{im}(f)$, $\quad y_2 = f(x_2) = f(x_2')$, so $y_2 \in \text{im}(f)$

$\text{im}(f) = (-\infty, R]$

Definition Vector spaces V and W are **isomorphic** if there is a linear trans $T: V \to W$ which is one-to-one and onto. Such a linear transformation is called an **isomorphism**.

Ex 1) (yesterday) $\mathbb{P}_n$ and $\mathbb{R}^{n+1}$ are isomorphic.

2) $\mathbb{P}_3$ and M_{22} are isomorphic.

Define $T: \mathbb{P}_3 \to M_{22}$ by

$$T(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ T(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ T(x^2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$T(x^3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

~~T(ab~~

~~T(a_0 + a_1x + a_2x~~

T is onto: Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{22}$.

Then
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= aT(1) + bT(x) + cT(x^2) + dT(x^3)$$
$$= T(a + bx + cx^2 + dx^3)$$

Thus T is onto M_{22}.

$T: V \to W$ $\quad$ T is <u>onto</u> if $\mathrm{im}(T) = W$

For all $w \in W$ there is $v \in V$ so that
$w = T(v)$.

Pick any $w \in W$. ~~Find $v \in V$ satisfy~~
Solve the equation $T(v) = w$ for v.

By dimension theorem,

$$\dim(\mathbb{P}_{\cancel{n}\,3}) = \dim(\mathrm{im}(T)) + \dim(\ker(T))$$

$\dim(\mathbb{P}_{\cancel{n}\,3}) = \cancel{n+1}.\ 4$

$\mathrm{im}(T) = M_{22}$, so $\dim(\mathrm{im}(T)) = 4$

So $\quad 4 = 4 + \dim(\ker(T))$,

thus $\ker(T) = \{0\}$, hence T is one-to-one.

3) $\mathbb{P}_4$ and M_{22} are not isomorphic.

Suppose $T: \mathbb{P}_4 \to M_{22}$ a linear transformation.

Dimension theorem:

$$\dim(\mathbb{P}_4) = \dim(\operatorname{im}(T)) + \dim(\ker(T)).$$

$\dim(\mathbb{P}_4) = 5$

$\operatorname{im}(T)$ is a subspace of M_{22}

so $\dim(\operatorname{im}(T)) \leq 4$

Thus $\dim(\ker(T)) \geq 1$. Thus $\ker(T) \neq \{0\}$.

Thus T is not one-to-one, and hence not an isomorphism.

Theorem Let V, W be finite-dimensional vector spaces, $T: V \to W$ a linear transformation. Then the following are equivalent:

1) T is an isomorphism
2) If $\{e_1, \dots, e_n\}$ is any basis of V then $\{T(e_1), \dots, T(e_n)\}$ is a basis of W
3) There is a basis $\{e_1 \dots e_n\}$ of V s.t. $\{T(e_1) \dots, T(e_n)\}$ is a basis of W.

Corollary V, W fin. dim. vector spaces. Then V, W are isomorphic if and only if $\dim(V) = \dim(W)$.