Lecture 14 Dimension Theorem

$T: V \to W$

$\ker(T) = \{0_V\}$ "there is only one element that T sends to 0_W"

If $\ker(T) = \{0\}$ then T is one-to-one.

__Theorem__ Let $T: V \to W$ be a linear transformation, and let $\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_n\}$ be a basis for V s.t. $\{e_{r+1}, \ldots, e_n\}$ is a basis for $\ker(T)$. Then $\{T(e_1), \ldots, T(e_r)\}$ is a basis for $\operatorname{im}(T)$. In particular,

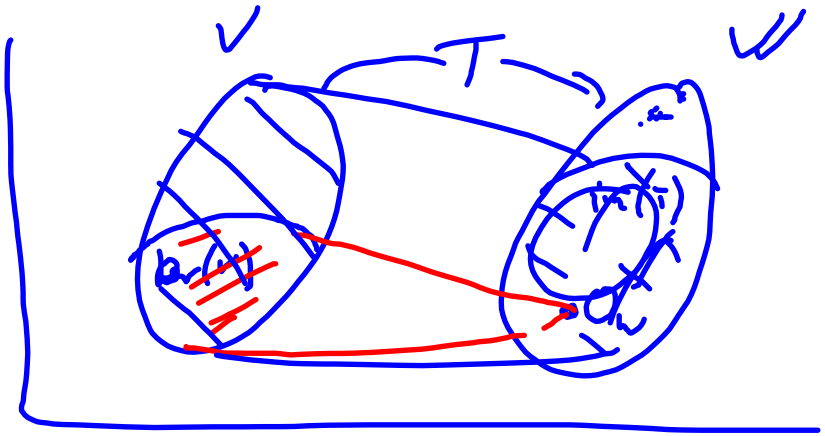
$$\dim(V) = r + n - r = \cancel{\dim(\ker(T))}) +$$

$$\underline{\dim(V) = \dim(\operatorname{im}(T)) + \dim(\ker(T))}$$

"rank-nullity theorem"

Proof We need to show that

$\{T(e_1), \ldots, T(e_r)\}$ is lin. ind. and spans $\mathrm{im}(T)$.



Consider $w \in \mathrm{im}(T)$.

Then $w = T(v)$ for some $v \in V$. So

$$v = a_1 e_1 + \cdots + a_n e_n \quad \text{for some } a_i \in \mathbb{R}$$

Then $w = T(v) = T(a_1 e_1 + \cdots + a_n e_n)$

$$= a_1 T(e_1) + \cdots + a_n T(e_n)$$

$$w = a_1 T(e_1) + \cdots + a_r T(e_r) + 0 + \cdots + 0$$

as $e_{r+1}, \ldots, e_n \in \ker(T)$.

Thus $w \in \mathrm{span}\{T(e_1), \ldots, T(e_r)\}$.

Suppose $s_1 T(e_1) + \dots + s_r T(e_r) = 0$

Then $T(s_1 e_1 + \dots + s_r e_r) = 0$

That is, $s_1 e_1 + \dots + s_r e_r \in \ker(T)$.

i.e. $s_1 e_1 + \dots + s_r e_r \in \operatorname{span}\{e_{r+1}, \dots, e_n\}$

Since $\{e_1, \dots, e_r, e_{r+1}, \dots, e_n\}$ is independent,

$s_1 e_1 + \dots + s_r e_r = 0$.

Since $\{e_1, \dots, e_r\}$ is ind, $s_1 = \dots = s_r = 0$.

Thus $\{T(e_1), \dots, T(e_r)\}$ is independent. $\square$

Ex. $T : \mathbb{P}_5 \longrightarrow \mathbb{P}_5$

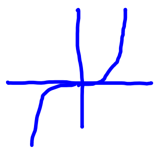
$$T(p(x)) = p(x) + p(-x).$$

Easy to see that T is linear.

$$\ker(T) = \{p(x) : p(x) + p(-x) = 0\}$$

$$= \{p(x) : p(x) = -p(-x)\}$$

$=$ odd polynomials of degree ≤ 5.



Let $p(x) = a_0 + a_1 x + \dots + a_5 x^5 \in \ker(T)$.

Then $a_0 + \dots + a_5 x^5 = -(a_0 - a_1 x + a_2 x^2 - a_3 x^3 + a_4 x^4 - a_5 x^5)$

Comparing coefficients of powers of x:

$a_0 = a_2 = a_4 = 0$, a_1, a_3, a_5 anything.

$\ker(T) = \{a_1 x + a_3 x^3 + a_5 x^5 : a_1, a_3, a_5 \in \mathbb{R}\}$

basis for $\ker(T)$ is $\{x, x^3, x^5\}$

Dimension theorem: $\dim(\mathbb{P}_5) = 6$,

$\dim(\ker(T)) = 3$, so $\dim(\operatorname{im}(T)) = 3$.

Also a basis for $\operatorname{im}(T) = \{T(1), T(x^2), T(x^4)\}$

$= \{2, 2x^2, 2x^4\}$

$\operatorname{im}(T) = \{$ even polynomials of deg $\leq 5\}$.

$\operatorname{im}(T) = \{q(x) \in \mathbb{P}_5 :$ there is $p(x)$ s.t.

$q(x) = p(x) + p(-x)\}$

Title: Oct 9 - 1:03 PM (4 of 5)

Ex

Consider $\overset{\text{lin trans}}{\vee}$ $T : \mathbb{P}_n \longrightarrow \mathbb{R}^{n+1}$ defined by

$$T(1) = e_1, \quad T(x) = e_2, \ldots, \quad T(x^n) = e_{n+1},$$

where $\{e_1, \ldots, e_{n+1}\}$ is the standard basis for $\mathbb{R}^{n+1}$.

$$T(a_0 + a_1 x + \ldots + a_n x^n) = a_0 T(1) + \ldots + a_n T(x^n)$$

$$= a_0 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + a_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}$$

T is clearly onto $\mathbb{R}^{n+1}$.

$$\dim(\mathbb{P}_n) = \dim(\operatorname{im}(T)) + \dim(\ker(T))$$

$$n+1 = n+1 + \dim(\ker(T))$$

So $\dim(\ker(T)) = 0$, i.e. $\ker(T) = \{0\}$

Thus T is one-to-one.