Lecture 12: Linear transformations: kernel and image

Ex. Define $T: P_3 \longrightarrow \mathbb{R}^4$ to be a linear transformation such that

$$T(1) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad T(x) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad T(x^2) = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \quad T(x^3) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Then $T(p(x)) = T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) =$

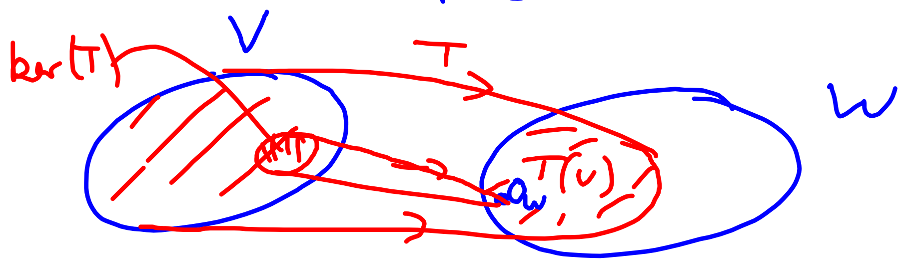
$$= a_0 T(1) + a_1 T(x) + a_2 T(x^2) + a_3 T(x^3)$$

$$= \begin{pmatrix}0\\ \vdots\\ 1\end{pmatrix} + \begin{pmatrix}0\\a_1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\ \vdots\\ 1\end{pmatrix} + a_3 \begin{pmatrix}0\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}0\\a_1\\0\\a_3\end{pmatrix}$$

Defn. Let $T: V \to W$ be a linear trans.

Kernel of T, $\ker(T) = \{ v \in V : T(v) = \bar{0}_W \}$

Image of T, $\operatorname{im}(T) = \{ w \in W :$ there exists a $v \in V$ s.t. $T(\bar{v}) = \bar{w} \} = T(V)$



Proposition $\ker(T)$ is a subspace of V

$\operatorname{im}(T)$ is a subspace of W.

Pf. $T(\bar{0}_V) = T(\bar{0}_V + \bar{0}_V) = T(\bar{0}_V) + T(\bar{0}_V)$ by T1

By cancellation, $T(0_V) = 0_W$.

Thus $0_V \in \ker(T)$.

[Also $0_W \in \operatorname{im}(T)$.]

Let $v_1, v_2 \in \ker(T)$.

$$T(v_1+v_2) = T(v_1) + T(v_2) \quad \text{by T1}$$
$$= \bar{0}_W + \bar{0}_W$$
$$= \bar{0}_W$$

Thus $v_1 + v_2 \in \ker(T)$.

Let $r \in \mathbb{R}$ $\quad T(rv_1) = rT(v_1)$ by T2

$$= r\bar{0}_W = \bar{0}_W$$

So $rv_1 \in \ker(T)$

$\boxtimes$

<u>Ex</u>. $T: \mathbb{P}_3 \to \mathbb{R}_4$ as before.

$\ker(T) = \text{span}\{1, x^2\}$

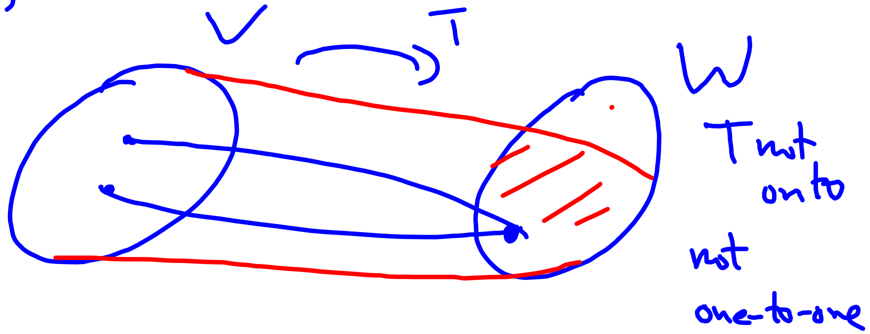
$\text{im}(T) = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 : a = c = 0 \right\}$

$\dim(\text{im}(T)) = 2 \qquad \text{im}(T)$ is a proper subspace of $\mathbb{R}^4$

$\dim(\ker(T)) = 2 \qquad \ker(T)$ is non-trivial

Note: $\underbrace{\dim(\text{im}(T))}_{\text{rank of } T} + \underbrace{\dim(\ker(T))}_{\text{nullity of } T} = \dim(\mathbb{P}_3)$.

Defn Let $T: V \to W$ be a linear transformation. We say T is onto or surjective if $\mathrm{im}(T) = W$. We say T is one-to-one or injective if for every $v_1, v_2 \in V$, if $T(v_1) = T(v_2)$ then $v_1 = v_2$.



Thm T is one-to-one if and only if $\dim(\ker(T)) = 0$.

Pf. Suppose T is one-to-one. ~~Then~~
Since $T(\bar{0}_V) = 0_W$, if $T(\bar{v}) = 0_W$ then $\bar{v} = \bar{0}_V$. So $\ker(T) = \{\bar{0}_V\}$ i.e. $\dim(\ker(T)) = 0$.

Suppose $\dim(\ker(T)) = 0$; i.e. $\ker(T) = \{\bar{0}_V\}$.

To show one-to-one, let $T(v_1) = T(v_2)$.

Then $T(v_1) - T(v_2) = \bar{0}_W$

$T(v_1 - v_2) = \bar{0}_W$ by T1, T2.

i.e. $v_1 - v_2 \in \ker(T)$.

i.e. $v_1 - v_2 = \bar{0}_V$

$v_1 = v_2$, as required. $\boxtimes$

Ex. A an $m \times n$ matrix. The function

$T_A(X) = AX$ from $\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

$\ker(T_A) = \{X \in \mathbb{R}^n : AX = 0\}$

The eqn $AX = 0$ has non-trivial solutions if and only if A is not invertible.

$\dim(\ker T_A) \neq 0 \iff \det(A) = 0 \iff A$ not invertible

$\iff T_A$ is not 1-1.