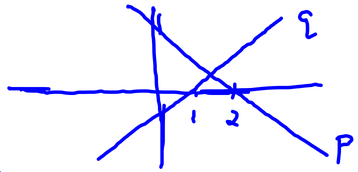
§64 20/.

$p, q \in \mathbb{P}_1$ $\quad p(1) \neq 0, \; q(2) \neq 0$

$p(2) = 0, \; q(1) = 0.$

Show $\{p, q\}$ is a basis for $\mathbb{P}_1$



$\dim(\mathbb{P}_1) = 2$

If $\{p, q\}$ is ind, then it is a spanning set.

To show $\{p, q\}$ is indep, suppose

$$r_1 p + r_2 q = 0. \quad (1)$$

Evaluate (1) at $x = 1$: $\quad r_1 p(1) + r_2 q(1) = \bar{0}(1)$

$$r_1 p(1) + \cancel{r_2 0} = 0$$

Since $p(1) \neq 0$, $r_1 = 0$.

Eqn (1) becomes $r_2 q = 0$.
Since $q \not\equiv 0$ $(q(2) \neq 0)$, this eqn says $r_2 = 0$.

(b) $\underbrace{\{p_0, \ldots, p_n\}}_{n+1}$ polys in $\underbrace{\mathbb{P}_n}_{\dim(\mathbb{P}_n) = n+1}$

$\exists\, a_0, \ldots, a_n$ all diff st. $p_i(a_i) \neq 0$,

$p_j(a_i) = 0$

Lecture 11: Linear Transformations

V, W vector spaces, $T: V \to W$ a function. T is a linear transformation if

T1 $\forall v_1, v_2 \in V$ $T(v_1+v_2) = T(v_1)+T(v_2)$

T2 $\forall v \in V\ \forall a \in \mathbb{R}$ $T(av) = aT(v)$.

<u>Exs</u> 1) $T: M_{nn} \to \mathbb{R}$

$$T(A) = tr(A) = tr(a_{ij}) = \sum_{i=1}^{n} a_{ii}$$

T is a linear transformation.

<u>T1</u> Let $A = (a_{ij})$, $B = (b_{ij}) \in M_{nn}$.

$$T(A+B) = T((a_{ij}+b_{ij})) = \sum_{i=1}^{n} a_{ii}+b_{ii}$$

$$T(A)+T(B) = \sum a_{ii} + \sum b_{ii} = \sum_{i=1}^{n}(a_{ii}+b_{ii})$$

This verifies T1.

T2 Let $r \in \mathbb{R}$.

$$T(rA) = T((ra_{ij})) = \sum ra_{ii} = r\sum a_{ii}$$
$$= r\,T(A).$$

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2) $S : M_{nn} \to \mathbb{R}$

$S(A) = \det(A)$

S is not linear.

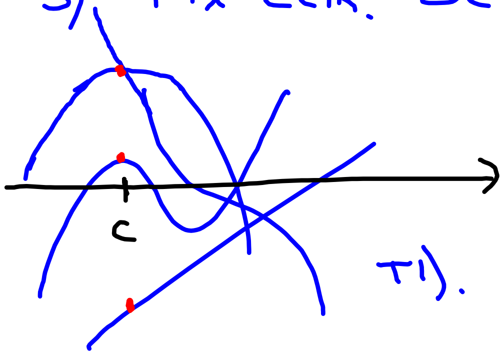
$S(A+B) = \det(A+B) \neq \det(A) + \det(B)$ in general.

$S(rA) = \det(rA) = r^n \det(A) \neq r\det(A)$

in general.

3) Fix $c \in \mathbb{R}$. Define $E_c : \mathbb{P}_3 \to \mathbb{R}$

$E_c(p(x)) = p(c)$.



E_c is a linear trans.

T1). $E_c(p(x)+q(x)) = E_c((p+q)(x))$

$= (p+q)(c)$

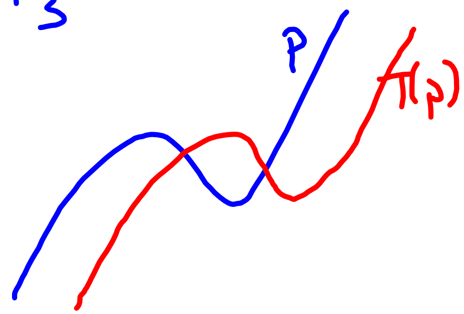
$= p(c) + q(c)$

$= E_c(p) + E_c(q)$

T2) similarly.

4) Define $T : \mathbb{P}_3 \to \mathbb{P}_3$

$$T(p(x)) = p(x-c)$$



T is linear:

T1. $T(p+q) = (p+q)(x-c)$
$= p(x-c) + q(x-c) = T(p) + T(q)$.

T2 $T(ap) = (ap)(x-c) = a\,p(x-c)$
$= a\,T(p)$.

<u>Theorem</u> Let V, W be vector spaces, $\{v_1, \ldots, v_n\}$ a basis for V and $w_1, \ldots, w_n$ any vectors in W. Then there is a unique linear transformation $T : V \to W$ with the property that $T(v_i) = w_i$ for $i = 1, \ldots, n$.

Proof. Define a ~~ts~~ function $T: V \to W$ as follows. Given any $v \in V$, we can write $v = r_1 v_1 + \dots + r_n v_n$ uniquely.

Define $T(v) = r_1 w_1 + r_2 w_2 + \dots + r_n w_n$.

T has the property:

$$T(v_i) = T(0v_1 + \dots 1v_i + \dots 0v_n)$$
$$= 0w_1 + \dots + 1w_i + \dots + 0w_n = w_i$$

T is linear:

T1 Let $v, u \in V$

$$T(v+u) = T(r_1 v_1 + \dots + r_n v_n + s_1 v_1 + \dots + s_n v_n)$$
$$= T((r_1 + s_1)v_1 + \dots + (r_n + s_n)v_n)$$
$$= (r_1 + s_1)w_1 + \dots + (r_n + s_n)w_n$$
$$= r_1 w_1 + \dots + r_n w_n + s_1 w_1 + \dots + s_n w_n$$
$$= T(v) + T(u).$$

T2: similarly.

T is unique:

Suppose $S: V \to W$ linear, st. also

$$S(v_i) = w_i.$$

Let $v \in V$.

$$S(v) = S(r_1 v_1 + \cdots + r_n v_n)$$

$$= r_1 S(v_1) + \cdots + r_n S(v_n) \quad \text{as } S \text{ is linear}$$

$$= r_1 w_1 + \cdots + r_n w_n$$

$$= T(v).$$

Thus $S = T$. ∎