Lecture 10.

Defn. Let U, W be subspaces of the vector space V. Then

$U \cap W = \{ v : v \text{ is in both } U \text{ and } W \}$

intersection

$U + W = \{ u + w : u \in U, w \in W \}$.

vector sum

Theorem

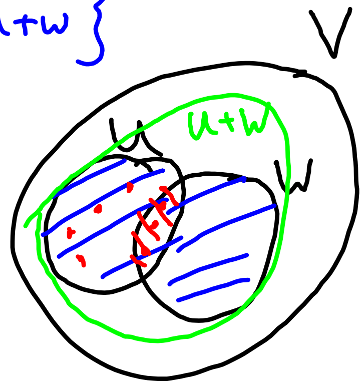
1) $U \cap W$ and $U + W$ are both vector ~~su~~ subspaces of V.

2) $\dim(U+W) = \dim(U) + \dim(\cancel{U}\,W) - \dim(U \cap \cancel{U}\,W)$.

Proof. $U \cap W$ is a subspace — exercise.

Show $U+W$ is a subspace.

$U+W = \{ v \in V :$ there exist $u \in U$ and $w \in W$ s.t. $v = u+w \}$



// $U \cup W$

// $U \cap W$

$\bar{0} = \bar{0} + \bar{0}$ and

$\bar{0} \in U$, $\bar{0} \in W$

Thus $\bar{0} \in U+W$.

Let $v_1, v_2 \in U+W$.

Then $v_1 = u_1 + w_1$

$v_2 = u_2 + w_2$

So $v_1 + v_2 = (u_1 + w_1) + (u_2 + w_2)$ using assoc. comm.

$= (u_1 + u_2) + (w_1 + w_2)$

$\in U + W$

Let $a \in \mathbb{R}$. $a v_1 = a(u_1 + w_1)$

$= a u_1 + a w_1$

$\in U + W$.

Thus $U+W$ is a subspace.

Let $\{x_1, \dots, x_k\}$ be a basis for $U \cap W$.

Extend $\{x_1, \dots x_k\}$ to a basis for U:
$\{x_1, \dots, x_k, u_1, \dots u_m\}$; and to a basis for
W: $\{x_1, \dots, x_k, w_1, \dots, w_n\}$.

<u>Claim</u> $\{x_1, \dots, x_k, u_1, \dots, u_m, w_1, \dots, w_m\}$ is
a basis for $U+W$.

Spanning set: Let $u+w \in U+W$

$$u+w = a_1x_1 + \dots + a_kx_k + b_1u_1 + \dots + b_m u_m$$
$$+ c_1x_1 + \dots + c_kx_k + d_1w_1 + \dots + d_nw_n$$
$$= (a_1+c_1)x_1 + \dots + (a_k+c_k)x_k + b_1u_1 + \dots$$
$$+ b_mu_m + d_1w_1 + \dots + d_nw_n$$

Thus $u+w \in \operatorname{Span}\{x_1 \dots u_1 \dots w_1 \dots\}$.

independence: Suppose

$$a_1x_1 + \dots + a_kx_k + b_1u_1 + \dots + b_mu_m + c_1w_1 + \dots + c_nw_n = \bar{0}$$

$$\underbrace{a_1x_1 + \dots + b_1u_1 + \dots}_{\in U} = \underbrace{-c_1w_1 - \dots - c_nw_n}_{\in W}$$

Thus $-c_1w_1 - \dots - c_nw_n \in U \cap W$, hence is in the span of $x_1, \dots, x_k$. But $\{x_1, \dots, w_1 \dots\}$ is independent, $-c_1w_1 - \dots - c_nw_n = \bar{0}$.

Since also $\{w_1 \dots w_n\}$ is indep, $c_1 = \dots = c_n = 0$.

Now, we have

$$a_1x_1 + \dots + b_1u_1 + \dots = \bar{0};$$

since $\{x_1, \dots, u_1 \dots\}$ is indep, we get

$a_1 = \dots = a_k = b_1 = \dots = b_m = 0$.

Thus $\{x_1 \dots x_k, u_1 \dots u_m, w_1 \dots w_n\}$ is indep.

Thus $\dim(U+W) = k + m + n$

$$\dim(U) + \dim(W) - \dim(U \cap W) = k+m+k+n-k$$
$$= k+m+n.$$

∎

Linear Transformations.

<u>Defn</u>. Let T be a function from U to V, U, V vector spaces. We say that T is a <u>linear</u> <u>transformation</u> if:

1) for all $u_1, u_2 \in U$, $T(u_1+u_2) = T(u_1)+T(u_2)$
2) for all $u_1 \in U, a \in \mathbb{R}$, $T(au_1) = aT(u_1)$.

Image of T:

Define: $\operatorname{im}(T) = T(U) = \{T(u) : u \in U\}$

$= \{v \in V : \text{there is } u \in U \text{ s.t. } T(u) = v\}$

$f : \mathbb{R} \to \mathbb{R}$ $\quad \operatorname{im}(f) = [0, \infty)$.

$f(x) = x^2$

Proposition Let $T: \mathcal{U} \to V$ be a lin. trans.
Then $\text{im}(T)$ is a subspace of V.

Pf. Easy to show that $T(\bar{0}_u) = \bar{0}_v$.

Thus $\bar{0}_v \in \text{im}(T)$.

Let $v_1, v_2 \in \text{im}(T)$. Then

$v_1 = T(u_1)$, $v_2 = T(u_2)$ $u_1, u_2 \in \mathcal{U}$.

Then $v_1 + v_2 = T(u_1) + T(u_2)$

$= T(u_1 + u_2)$ T1

Thus $v_1 + v_2 \in \text{Im}(T)$.