McMaster University Math 1XX3 Winter 2013
Midterm 1 — Practice Version

Duration: 60 minutes

Instructor: Dr. D. Haskell

Name: Solutions - DH

Student ID Number: ______________

This test paper is printed on both sides of the page. There are 7 question on 5 pages, with a blank page at the end for rough work. You are responsible for ensuring that your copy of this test is complete. Bring any discrepancies to the attention of the invigilator.

Instructions

(1) Only the standard McMaster calculator is allowed.
(2) All answers must be written in the space following the question. If you need more space, use the blank page at the end of the exam, and indicate clearly where to find the answer.
(3) Additional scratch paper is available for rough work; ask the invigilator.

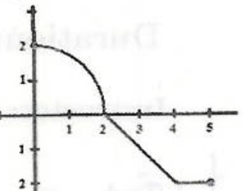
This PRACTICE version of the midterm is intended to give you an idea of the format, approximate length and approximate difficulty of the actual midterm. There is no guarantee as to the actual length and difficulty of the actual exam. In particular, the actual midterm will NOT be "just the same with the numbers changed".

Problem	Points
1 [10]	
2 [5]	
3 [5]	
4 [5]	
5 [5]	
6 [5]	
7 [5]	
Total [40]	

1) [10 points]

a) Let $f(x)$ be the function defined below, whose graph is shown.

$$f(x) = \begin{cases} \sqrt{4-x^2}, & \text{if } 0 \le x < 2; \\ 2-x, & \text{if } 2 \le x \le 4; \\ -2, & \text{if } 4 < x \le 5. \end{cases}$$



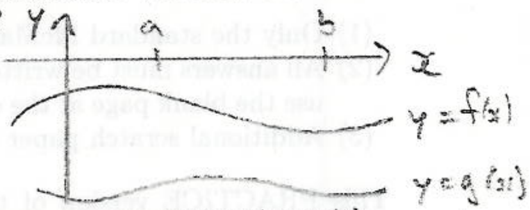
Calculate $\int_0^5 f(x)\,dx$.

$$\int_0^5 f(x)\,dx = \int_0^2 f(x)dx + \int_2^4 f(x)dx + \int_4^5 f(x)dx$$

$$= \text{area } \square - \text{area } \triangledown - \text{area } \square$$

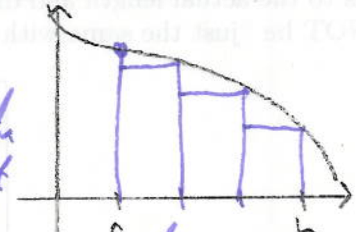
$$= \frac{1}{4}\pi 2^2 - \frac{1}{2}\cdot 2\cdot 2 - 1\cdot 2 = \pi - 4.$$

b) The graphs of $y = f(x)$ and $y = g(x)$ are shown. What calculation will find the area bounded by the graphs and the vertical lines $x = a$ and $x = b$?



$$\text{area} = \int_a^b (f(x) - g(x))\,dx$$

c) For the function whose graph is shown, decide if the Riemann sum approximation to the integral with sample points taken to be right endpoints is an overestimate or an underestimate to the exact area, and justify your answer.



underestimate

Because the function is decreasing, the rectangles with height given by a right endpoint lie below the curve.

d) Recall the formula $\sum_{i=5}^{n} i = \frac{1}{2}n(n+1)$. Find $\sum_{i=5}^{50} i$.

$$\sum_{i=5}^{50} i = \sum_{i=1}^{50} i - \sum_{i=1}^{4} i = \frac{1}{2}50(51) - \frac{1}{2}4(5)$$

e) Let $g(x) = \int_{-x}^{x} f(t)dt$. Find $g'(x)$.

$$g(x) = -\int_0^{-x} f(t)dt + \int_0^{x} f(t)dt.$$

By FTC I, $g'(x) = -f(-x)(-1) + f(x)$.

$$= f(-x) + f(x)$$

4) [5 points] Calculate the integral $\int \tan^3(x)\sec^2(x)\,dx$.

Let $u = \tan(x)$
$du = \sec^2(x)dx$

~~tan~~ $\int \tan^3(x)\sec^2(x)dx = \int u^3\,du$

$= \frac{1}{3}u^4 + C$

$= \frac{1}{3}\tan^4(x) + C$

5) [5 points] Calculate the improper integral $\int_0^1 x\sqrt{1-x^2}\,dx$ and state whether it converges or diverges.

$\int x\sqrt{1-x^2}\,dx = \int -\frac{1}{2}\sqrt{u}\,du$

$u = 1-x^2$
$du = -2x\,dx$
$-\frac{1}{2}du = x\,dx$

$= -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C$

$= -\frac{1}{2}(1-x^2)^{3/2} + C$

$\int_0^1 x\sqrt{1-x^2}\,dx = \left[-\frac{1}{2}(1-x^2)^{3/2}\right]_0^1$

$= -\frac{1}{2}(0) - \left(-\frac{1}{2}\right)(1-0)^{3/2}$

$= 1/2$

Note that this integral is <u>not</u> improper.

2) 5 points] Let $f(x) = \int_0^x (1-t^2)e^{t^2}\,dt$. Find the intervals on which f is increasing, and the intervals on which f is decreasing.

Find where $f' > 0$ and $f' < 0$. By FTC I, $f'(x) = (1-x^2)e^{x^2}$. As $e^{x^2} > 0$, solve $1 - x^2 = 0$
$x = \pm 1$.

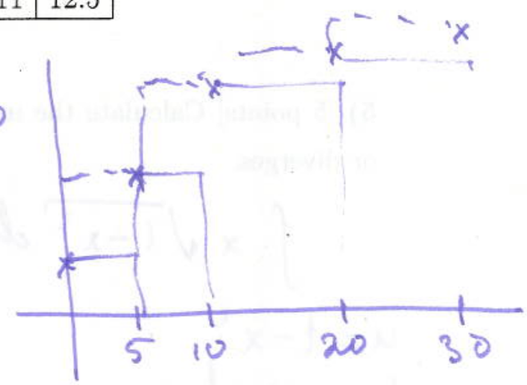
thus $f'(x) < 0$ so f is decreasing on $(-\infty; 1) \cup (1, \infty)$
$f'(x) > 0$ so f is increasing on $(-1, 1)$.

3) [5 points] The following data represents the velocity of an airplane in the minutes after take-off until it reaches its cruising speed. We want to estimate the distance travelled by the airplane in these 30 minutes.

time in minutes	0	5	10	20	30
velocity in kilometers per minute	3	6.5	9	11	12.5

a) Estimate the distance using lefthand endpoints.

left approx $= f(0)\,5 + f(5)\,5 + f(10)\cdot 10 + f(20)\cdot 10$
$= 3 + 16.25 + 90 + 110$

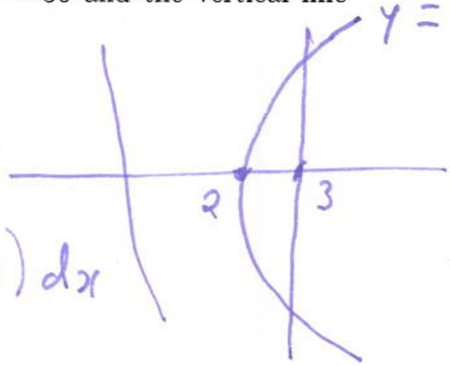


b) Estimate the distance using righthand endpoints.

right approx $= f(5)\cdot 5 + f(10)\,5 + f(20)\cdot 10 + f(30)\cdot 10$
$= 16.25 + 90 + 110 + 125.$

6) [5 points] Find the area bounded by the hyperbola $9x^2 - 4y^2 = 36$ and the vertical line $x = 3$.

$y^2 = \frac{9x^2 - 36}{4}$



when $y = 0$, $9x^2 - 4(0)^2 = 36$

$x = \pm 2$

$$\text{area} = \int_2^3 \sqrt{\tfrac{1}{4}(9x^2-36)} - \left(-\sqrt{\tfrac{1}{4}(9x^2-36)}\right) dx$$

$$= \int_2^3 \sqrt{9x^2-36}\, dx = \int_2^3 3\sqrt{x^2-4}\, dx$$

7) [5 points] Find the volume of the solid formed by rotating the region in problem 6) around the x-axis.

$$\text{volume} = \int_2^3 \pi \left(\sqrt{\tfrac{1}{4}(9x^2-36)}\right)^2 dx$$

$$= \int_2^3 \frac{\pi}{4}(9x^2-36)\, dx$$

$$= \frac{\pi}{4}\left[3x^3 - 36x\right]_2^3$$

$$= \frac{\pi}{4}\left((\cancel{27}\,81 - 108) - (\cancel{12}\,24 - 72)\right)$$