McMaster University Math 1XX3 Winter 2013
Midterm 1
February 4 2013

Duration: 60 minutes

Instructor: Dr. D. Haskell

Name: Solutions

Student ID Number: ____________

This test paper is printed on both sides of the page. There are 7 question on 5 pages, with a blank page at the end for rough work. You are responsible for ensuring that your copy of this test is complete. Bring any discrepancies to the attention of the invigilator.

Instructions

(1) Only the standard McMaster calculator is allowed.
(2) All answers must be written in the space following the question. If you need more space, use the blank page at the end of the exam, and indicate clearly where to find the answer.
(3) Additional scratch paper is available for rough work; ask the invigilator.

Problem	Points
1 [10]	
2 [5]	
3 [5]	
4 [5]	
5 [5]	
6 [5]	
7 [5]	
Total [40]	

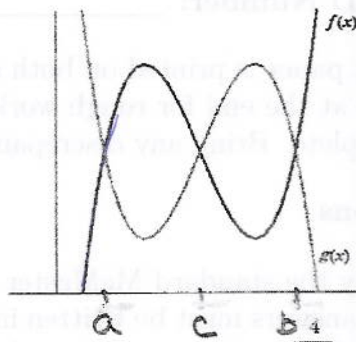
1) [10 points]

a) Calculate $\sum_{i=3}^{100} 1.$ $= \sum_{q=1}^{98} 1 = 98$

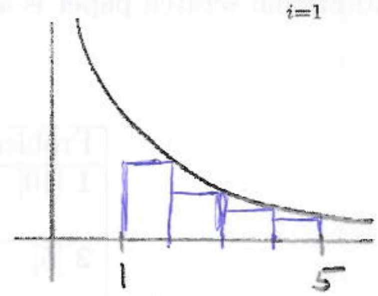
$q = i-2$

b) The graphs of the functions $y = f(x)$ and $y = g(x)$ are shown. What calculation will find the area enclosed by the curves?

$\int_a^c f(x)-g(x)dx + \int_c^b (g(x)-f(x))dx$



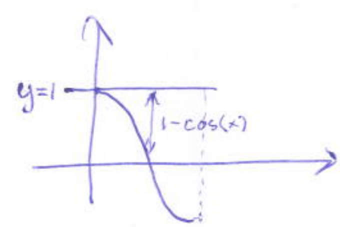
c) On the graph of $y = f(x)$ given, sketch the area calculated by the Riemann sum $\sum_{i=1}^{4} f(x_i)\Delta x$, where $\Delta x = \frac{5-1}{4}$ and $x_i = 1 + i\Delta x$.



d) The function $y = \cos(x)$ is rotated around the horizontal line $y = 1$ from $x = 0$ to $x = \pi$. Express the volume of the resulting solid as an integral.

$V = \int_{x=0}^{\pi} \pi r(x)^2 dx = \int_{x=0}^{\pi} \pi (1-\cos(x))^2 dx$

$r(x) = 1-\cos(x)$

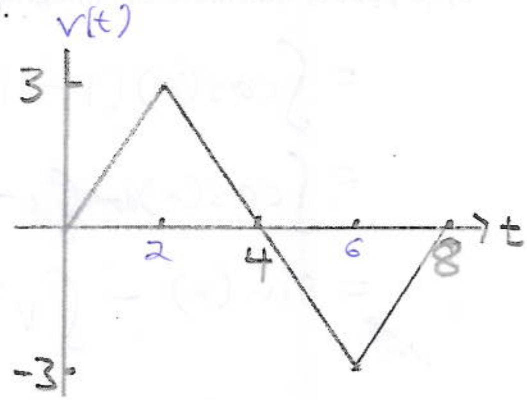


e) Let $g(x) = \int_1^{\sin(x)} \sqrt{1+t^2}dt$. Find $g'(x)$.

$F(x) = \int_1^x \sqrt{1+t^2}dt$ $\qquad g(x) = F(\sin(x))$

$F'(x) = \sqrt{1+x^2}$ $\qquad \Rightarrow \boxed{g'(x) = \cos(x)\sqrt{1+\sin^2(x)}}$

2) 5 points] A particle travelling in a straight line has velocity function $v(t)$ with graph as shown. Recall that the position function $s(t)$ satisfies $s'(t) = v(t)$.



a) Find $s(4)$ and $s(8)$.

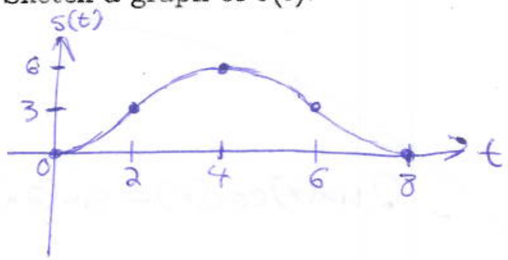
$$v(t) = \begin{cases} \frac{3}{2}t & \text{if } t\in[0,2) \\ -\frac{3}{2}t+6 & \text{if } t\in[2,6) \\ \frac{3}{2}t-12 & \text{if } t\in[6,8] \end{cases}$$

$$\boxed{s(4)=2\cdot 3=6}$$

$$\boxed{s(8)=0}$$

~~s(8)=0 and s(~~

b) Sketch a graph of $s(t)$.



c) Find the total distance travelled by the particle at time 8.

total distance $\Rightarrow \int_0^8 |s(t)|\,dt = 6+6=12$

3) [5 points] Calculate the improper integral $\displaystyle\int_e^\infty \frac{1}{x(\ln(x))^2}\,dx$ and state whether the integral converges or diverges. (Hint: substitute $u = \ln(x)$.)

$$\int_e^\infty \frac{1}{x(\ln(x))^2}\,dx = \int_{u=1}^\infty \frac{1}{u^2}\,du = \frac{-1}{u}\Big|_{u=1}^\infty \overset{\text{take lim}}{=} \lim_{t\to\infty}\left(\frac{-1}{t}+1\right) = 1$$

$u=\ln(x)$

$\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$

converges

4) [5 points] Calculate the integral $\int \cos^3(x)\,dx$.

$$= \int \cos(x)(1-\sin^2(x))\,dx$$

$$v = \sin(x) \Rightarrow \frac{dv}{dx} = \cos(x)$$

$$= \int \cos(x)\,dx - \int \cos(x)\sin^2(x)\,dx$$

$$= \sin(x) - \int v^2\,dv = \sin(x) - \frac{v^3}{3} + C$$

$$= \sin(x) - \frac{\sin^3(x)}{3} + C$$

where $C \in \mathbb{R}$

$$2\sin(x)\cos(x) = \sin(2x)$$

5) [5 points] Calculate the integral $\int \sqrt{9-x^2}\,dx$.

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

$$\Rightarrow 2\cos^2(x) - 1 = \cos(2x)$$

$$x = 3\sin(v)$$

$$\Rightarrow dx = 3\cos(v)\,dv$$

$$\int \sqrt{9-x^2}\,dx = \int 3\underbrace{\sqrt{1-\sin^2(v)}}_{=\cos^2(v)}(3\cos(v))\,dv$$

$$= 9\int \cos^2(v)\,dv$$

$$\cos^2(v) = \frac{1}{2}(\cos(2x)+1)$$

$$= \frac{9}{2}\int(\cos(2v)+1)\,dv = \frac{9}{4}\sin(2v) + \frac{9}{2}v + C$$

$$= \cancel{\frac{9}{2}\left(\frac{1}{2}\sin(v)\cos(v)\right)} = \frac{9}{2}\left(\frac{\sin(2\arcsin(x/3))}{2} + \arcsin\left(\frac{x}{3}\right)\right) + C$$

$C \in \mathbb{R}$

6) [5 points] Find the volume of the solid formed by rotating the graph of $y = \sin(x)$ around the x-axis from $x = 0$ to $x = \pi$.

radius squared

$$V = \int_{x=0}^{\pi} \pi\left(\sin^2(x)\right)dx$$

$$1 - 2\sin^2(x) = \cos(2x) \Rightarrow \sin^2(x) = \frac{-\cos(2x)+1}{2}$$

$$= \pi\int_{x=0}^{\pi} \sin^2(x)\,dx = \pi\int_{x=0}^{\pi} \frac{1-\cos(2x)}{2}\,dx$$

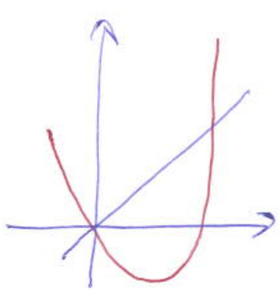
$$= \frac{\pi^2}{2} - \int_{x=0}^{\pi} \frac{\cos(2x)}{2}\,dx = \frac{\pi^2}{2} - \cancel{\frac{1}{2}\left(\frac{\sin(2x)}{4}\right)\Big|_{x=0}^{\pi}} \to 0$$

$$\boxed{V = \frac{\pi^2}{2}}$$

7) [5 points] Find the area bounded by the graphs of $y = 5x - x^2$ and $y = 3x$.

$$5x - x^2 = 3x \Rightarrow 2x - x^2 = 0 \Rightarrow x(2-x) = 0$$

$$\Rightarrow \boxed{x = 0} \text{ or } \boxed{x = 2}$$



$$\int_{x=0}^{2} \left((5x - x^2) - 3x\right)dx = \int_{x=0}^{2} (2x - x^2)\,dx$$

$$= x^2 - \frac{x^3}{3}\Big|_0^2 = 4 - \frac{8}{3} = \frac{12-8}{3} = \boxed{\frac{4}{3}}$$