# Math 1C03 Introduction to Mathematical Reasoning
# Term 2 Winter 2014–2015
# Problem Sheet 8: complex numbers
# to be completed by Monday March 16 2015

1) Review calculations with complex numbers in both cartesian and polar coordinates by doing problems selected from Exercise Set 8 on pages 218–221 of the text.

2) Let $z$, $w$ be any complex numbers.
   a) Prove that $|zw| = |z||w|$ (this is trivial when $z$ and $w$ are written in polar form; it requires a bit more algebra to show when they are given in cartesian coordinates).
   b) Give an example that shows that, in general, $|z + w| \neq |z| + |w|$.
   c) Describe $\{z : |z + 2| = |z| + 2\}$ as a subset of the complex plane.
   d) Describe $\{z : |z + 3i| = |z| + |3i|\}$ as a subset of the complex plane.
   e) Describe $\{z : |z + 2 + 3i| = |z| + |2 + 3i|\}$ as a subset of the complex plane.
   f) Generalise: describe $\{z : |z + u + vi| = |z| + |u + vi|\}$, for any complex number $w = u + vi$.

3) a) Describe $\{z \in \mathbb{C} : |z| = 10\}$.
   b) Describe $\{z \in \mathbb{C} : |z - 4| = 10\}$.
   c) Describe $\{z \in \mathbb{C} : |z - (4 + 3i)| = 10\}$.
   d) Generalise: describe $\{z \in \mathbb{C} : |z - (u + vi)| = r\}$, for any $u + vi \in \mathbb{C}$ and $r \in \mathbb{R}$.

4) The function $f(z) = \dfrac{az + b}{cz + d}$, where $a$, $b$, $c$, $d$ are complex numbers with $ad - bc \neq 0$ is called a *fractional linear transformation*.
   a) Show that $f$ is injective on its domain, which is $\{z \in \mathbb{C} : z \neq -d/c\}$.
   b) Find the range of $f$.
   c) A special case of a fractional linear transformation occurs when $a = 1$ and $c = 0$. Describe the effect of this fractional linear transformation geometrically and hence show that it maps a circle $\{z \in \mathbb{C} : |z - (u + vi)| = r\}$ to another circle of the same radius (what is the centre of this circle?).

5) Prove that the sum of the 5th roots of unity is equal to 0. Generalise.

6) The theory of integration of functions of a complex variable needs to be developed. Assume that it follows the same rules as integration of functions of a real variable, and calculate
$$\int e^{(a+ib)x}\ dx.$$
By considering real and imaginary parts, find $\int e^{ax}\cos(bx)\ dx$ and $\int e^{ax}\sin(bx)\ dx$. If you like, you can verify your answer by integrating by parts instead.