Math 1C03 Introduction to Mathematical Reasoning
Term 2 Winter 2014–2015
Problem Sheet 6: rationals as congruence classes, properties of functions to be completed by Monday March 2 2015

1) Recall that we defined the rational numbers formally as the set of congruence classes of pairs of integers (a, b), where $b \neq 0$:

$$[(a, b)] = \{(x, y) \in \mathbb{Z} \times \mathbb{Z}^{\neq 0} : ay = xb\}.$$

   i) Prove that the definition of multiplication of rational numbers:

$$[(a, b] \times [(c, d)] = [(ac, bd)]$$

   is *well-defined*. That is, if $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$ then $[(ac, bd)] = [(a'c', b'd')]$.

   ii) Consider the function $f : \mathbb{Q} \to \mathbb{Z}$ defined by $f([(a, b)]) = a+b$. Prove that f is not well-defined (and hence I should not have called it a function).

2) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = ax + b$, where a, b are any real numbers, $a \neq 0$. Prove that f is injective and surjective.

3) Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by

$$f(n) = \begin{cases} n + 5, & \text{if } n \text{ is even;} \\ n - 5, & \text{if } n \text{ is odd.} \end{cases}$$

   i) Calculate some values of f. Prove that, if $m \neq n$, then $f(m) \neq f(n)$ and hence f is injective (careful: you have three cases to consider).

   ii) For different possible output values (say, $10, 31, -1, -14$) what input values will give you these outputs? Prove that f is surjective (careful: you have to take an arbitrary integer x and find what integer n will map to it).

   iii) Since f is bijective it has an inverse. Express the inverse function explicitly.

4) i) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is injective but not surjective.

   ii) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is surjective but not injective.

   iii) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is neither injective nor surjective.

   iv) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is both injective and surjective.

   v) Change the domain and range and repeat.