Math 1C03 Introduction to Mathematical Reasoning
Term 2 Winter 2014–2015
Problem Sheet 5: strong induction, rational numbers and real numbers
to be completed by Monday February 23 2015
(enjoy your Reading Week)

1) The *Fibonacci sequence* is given by taking each element to be the sum of the preceding two. We can generalise this idea to get the so-called *Tribonacci sequence*: $T_1 = T_2 = T_3 = 1$, $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$. We want to show that for all $n \in \mathbb{Z}^+$, $T_n < 2^n$. Call this property $P(n)$.
   - i) Verify that $P(1)$, $P(2)$ and $P(3)$ all hold.
   - ii) Verify that $P(4)$ holds by using the results in (i). (This step is not strictly necessary, but is helpful for getting an idea of how to do the general step.)
   - iii) Assume that $k \geq 4$ and that $P(i)$ holds for all $i \leq k$. Prove that $P(k+1)$ holds.
   - iv) Deduce by strong induction that $P(n)$ holds for all $n \in \mathbb{Z}^+$. (Notice that you needed all the information from (i) in your base case.)

2)
   - i) Use Fermat's Little Theorem to prove that, for every prime p other than 2 or 5 there is some positive integer r such that $10^r \equiv 1 \mod p$.
   - ii) Use Fermat's Little Theorem twice to prove that, for every pair of distinct primes p and q other than 2 or 5, there is some positive integer r such that $10^r \equiv 1 \mod pq$.
   - iii) Use (i) and the Binomial Theorem to prove that for any $1 \leq i \leq p$, there is some positive integer r such that $10^r \equiv 1 \mod p^i$ (in fact, the same r works for all of these values of i). You will need the fact from Problem Sheet 4 that the binomial coefficient $\binom{p}{n}$ is divisible by p. Generalise to conclude that, for any power of p there is some positive integer r such that $10^r \equiv 1 \mod p^i$.
   - iv) Any positive integer n can be factored as a product of distinct primes, $n = p_1^{m_1} \ldots p_s^{m_s}$. Prove by induction on s that if $\gcd(n, 10) = 1$ then there is some positive integer r such that $10^r \equiv 1 \mod n$. (The notation here will get somewhat cumbersome; if you prefer, you can just do the case when $m_i = 1$ for all i.)
   - v) Look at the proof of Theorem 5.43 in the text. If the number $1/n$ has a periodic decimal expansion with period s, then the remainders r_i in the calculation will satisfy $r_{i+s} = r_i$ for every i. Show that the period s satisfies $10^s \equiv 1 \mod n$.
   - vi) Conclude that $1/n$ has a periodic decimal expansion with period s if and only if s is the least positive integer such that $10^s \equiv 1 \mod n$.